Soft-Collinear Gravity and Soft Theorems

软共线引力与软定理

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This chapter reviews the construction of ”soft-collinear gravity,” the effective field theory which describes the interaction of collinear and soft gravitons with matter (and themselves), to all orders in the soft-collinear power expansion, focusing on the essential concepts. Among them are an emergent soft background gauge symmetry, which lives on the light-like trajectories of energetic particles and allows for a manifestly gauge-invariant representation of the interactions in terms of a soft covariant derivative and the soft Riemann tensor, and a systematic treatment of collinear interactions, which are absent at leading power in gravity. The gravitational soft theorems are derived from soft-collinear gravity at the Lagrangian level. The symmetries of the effective theory provide a transparent explanation of why soft graviton emission is universal to sub-sub-leading power but gauge boson emission is not and suggest a physical interpretation of the form of the universal soft factors in terms of the charges corresponding to the soft symmetries. The power counting of soft-collinear gravity further provides an understanding of the structure of loop corrections to the soft theorems.

本章回顾了软对易引力的构建，这是一种描述共线引力子、软引力子与物质 (以及它们自身) 相互作用的有效场论，适用软对易幂展开的所有阶，本章聚焦核心概念。其中包括涌现的软背景规范对称性: 该对称性存在于高能粒子的类光轨迹上， 使得相互作用可以用软协变导数和软黎曼张量表示为明显规范不变的形式；另外还包括对引力中领头幂阶不存在的共线相互作用的系统处理。引力软定理可在拉格朗日层面从软对易引力导出。有效理论的对称性清晰解释了为何软引力子发射到次次领头幂阶仍具有普适性，而规范玻色子发射却不具备，并给出了普适软因子形式基于软对称性对应荷的物理解释。软对易引力的幂计数还进一步帮助我们理解软定理圈修正的结构。

Keywords

关键词

Gravitation - Soft-collinear effective field theory - Effective Lagrangian - Soft and collinear divergences - Soft theorem - Power corrections -

引力——软共线有效场论——有效拉格朗日量——软发散与共线发散——软定理——幂修正——

Einstein-Hilbert theory

爱因斯坦-希尔伯特理论

Introduction

引言

”My reasons for now attacking this question are: (1) Because I can. [...] (2) Because something might go wrong and this would be interesting. Unfortunately, nothing does go wrong.” S. Weinberg, Ref. [41]

“我如今着手研究这个问题的原因是:(1) 因为我有能力研究。……(2) 因为没准会出问题，那会很有意思。可惜，什么问题都没出。” S. 温伯格，参考文献 [41]

The gravitational force is widely perceived to be fundamentally different from the gauge forces that govern the other microscopic interactions of the elementary particles. The gravitational interactions are inevitably non-renormalizable, calling for a modification at very short distances (or a non-trivial ultraviolet fixed point). Their underlying gauge symmetry is related to space-time transformations, in contrast to the internal $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetries operating on fields in rigid Minkowski space-time.

人们普遍认为，引力与支配基本粒子其他微观相互作用的规范力存在本质区别。引力相互作用必然不可重整，因此需要在极短距离下对理论进行修正 (或是存在非平凡的紫外不动点)。引力的基本规范对称性与时空变换相关，这和作用于刚性闵氏时空中场的内 $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ 规范对称性不同。

Yet, from the low-energy perspective and applying the basic principles of quantum field theory, the Lagrangian of weak-field gravity on Minkowski space follows from the desire to construct a consistent theory for a massless spin-2 particle in very much the same way as gauge theories do for the case of a massless spin-1 particle. The universality and space-time symmetry of gravitation then arises from the requirement that interacting massless fields with spin larger than $\frac{1}{2}$ must couple to a conserved current, which is the energy-momentum tensor for spin-2. This motivates a closer inspection of the relation between gauge theory and gravitational scattering in Minkowski space.

但从低能视角出发，应用量子场论的基本原理，闵氏空间上弱场引力的拉格朗日量，是遵循量子场论基本原理构造得到的: 就像规范理论描述自旋 1 零质量粒子一样，引力的拉格朗日量是自旋 2 零质量粒子的自洽理论。引力的普适性与时空对称性源于如下要求: 自旋大于 $\frac{1}{2}$ 的相互作用零质量场必须耦合到守恒流，对于自旋 2 而言这个守恒流就是能量动量张量。这促使我们更细致地考察闵氏空间中规范理论与引力散射的关系。

The study of gravitational scattering amplitudes in quantized weak-field gravity has attracted much attention after the discovery of remarkable relations between graviton and gluon scattering amplitudes [13, 14], which state that tree-level amplitudes of the former can be obtained by ”squaring” Yang-Mills tree-level amplitudes and replacing color factors by kinematic ones. The simplest example is the three-point amplitude in spinor-helicity notation (reviewed in [28]):

在引力子与胶子散射振幅间的奇妙关系被发现后 [13,14]，量子化弱场引力中引力散射振幅的研究吸引了大量关注，该关系指出: 引力子的树图振幅可以通过将杨-米尔斯树图振幅“平方”，再将色因子替换为运动学因子得到。最简单的例子是旋量螺旋度记号下的三点振幅 (综述见 [28]):

$$\mathcal{A}_{\mathrm{YM}}\left(1_a^-, 2_b^-, 3_c^+\right) = -g_s f^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

(1)

$$\mathcal{A}_{\mathrm{EH}}\left(1^{--}, 2^{--}, 3^{++}\right) = \frac{\kappa}{2} \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}.$$

Numerous extensions of such ”double copy” or ”color-kinematics duality” relations have been found to different gauge/gravity theories, to non-trivial classical backgrounds, and to the one-loop level.

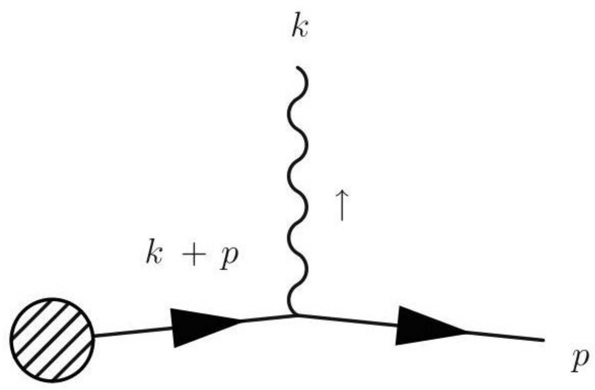
这类“双拷贝”即“颜色-运动学对偶”关系已经被发现有诸多推广，涵盖不同规范/引力理论、非平凡经典背景以及单圈阶。

Much can be learned by looking at the behavior of quantum amplitudes in the infrared (IR). When an energetic, massless particle with momentum p^μ emits another massless particle with momentum k^μ (see Fig. 1), the internal propagator $1/(p+k)^2$ becomes singular in the soft limit $k^\mu \to 0$ and in the collinear limit $p^\mu \parallel k^\mu$. When such configurations are integrated over the phase space of the emitted particle or appear inside loops, the result is a logarithmic divergence. It has been recognized from the early days of quantum field theory that the soft and collinear limits exhibit universal, process-independent features [17,29,31] and that a precise definition of quantum mechanical observables is required to obtain sensible, IR-finite results. The study of these limits in quantum electrodynamics and non-abelian gauge theories accordingly has a long history.

研究红外 (IR) 区域量子振幅的行为可以得到许多结论。当一个能动量为 p^μ 的高能零质量粒子辐射出另一个能动量为 k^μ 的零质量粒子时 (见图 1)，内部传播子 $1/(p+k)^2$ 在软极限 $k^\mu \to 0$ 与共线极限 $p^\mu \parallel k^\mu$ 下会变得奇异。对这类构型在辐射粒子的相空间积分后，或是这类构型出现在圈图内部时，结果会产生对数发散。量子场论发展早期人们就已经认识到，软极限和共线极限具有普适的、不依赖过程的性质 [17,29,31]，要得到合理的红外有限结果，需要对量子力学可观测量给出精确定义。因此，量子电动力学和非阿贝尔规范理论中对这些极限的研究由来已久。

Fig. 1 Emission of a boson with momentum k from an external energetic particle with momentum p

图 1 外能动量为 p 的高能粒子辐射出一个能动量为 k 的玻色子



In QCD, which is strongly interacting in the infrared, understanding the soft and collinear limit is of paramount importance to make predictions for high-energy scattering and culminates in powerful factorization theorems [24]. In this way, one isolates the infrared physics in some well-defined functions while leaving the more process-dependent features for perturbative calculations. Given the presence of several momentum scales, it is logical to apply effective Lagrangians to capture the IR physics, especially as Lagrangians are better suited than amplitudes to uncover the gauge invariance and recursive structure of multi-scale problems. Soft-collinear effective theory (SCET) [2, 3, 5, 6] has therefore emerged as an important conceptual and calculational tool for factorization in gauge theories. It is designed to precisely reproduce Feynman amplitudes in their soft and collinear limits. Moreover, at least in principle, it can do so beyond the leading power in the expansion in small-scale ratios. Although the case of high-energy gravitational scattering appears to be of less practical relevance, in view of the above-mentioned relations between graviton and Yang-Mills scattering amplitudes, which are presently not understood at Lagrangian level, it is suggestive to apply effective Lagrangian techniques to at least the soft and collinear limits of graviton amplitudes - which is the subject of this chapter.

在红外区强相互作用的量子色动力学中，理解软极限和共线极限对高能散射预言至关重要，其成果是强有力的因子化定理 [24]。通过这种方法，人们可将红外物理分离到定义明确的函数中，而将更依赖过程的特征留给微扰计算。由于存在多个动量标度，应用有效拉格朗日量来捕获红外物理是合乎逻辑的，尤其是拉格朗日量比振幅更适合揭示多尺度问题的规范不变性和递推结构。因此，软共线有效理论 (SCET) [2, 3, 5, 6] 已经成为规范理论因子化的重要概念与计算工具。它旨在精确重现软极限和共线极限下的费曼振幅。此外，至少在原则上，它可以在小标度比展开中得到超出领头幂次的结果。尽管高能引力散射看起来实际关联性较低，但鉴于上文提到的引力子和杨-米尔斯散射振幅之间的关联 (目前尚未在拉格朗日层面得到理解)，将有效拉格朗日技术应用于引力子振幅的软极限和共线极限是具有启发性的——这正是本章的主题。

These limits already exhibit interesting similarities and differences between massless spin-1 and spin-2 particles (gauge bosons and gravitons, respectively) coupled to matter. It has been noted long ago [41] that the "eikonal" or leading soft limit of gravity is very similar to gauge theory. The long-wavelength radiation "sees" only the direction of motion (classical trajectory) and charge of energetic particles. Thus, in the eikonal approximation, the amplitude for radiating a single soft graviton from energetic particles with momenta p_i^μ emerging from a hard scattering process,

这些极限已经展现出与物质耦合的无质量 1 自旋粒子 (规范玻色子) 和 2 自旋粒子 (引力子) 之间有趣的异同。很久以前人们就已经指出 [41]，引力的“程函”即领头软极限与规范理论非常相似。长波长辐射“观测”到的只有高能粒子的运动方向 (经典轨迹) 和荷。因此，在程函近似下，从硬散射过程出射的动量为 p_i^μ 的高能粒子辐射单个软引力子的振幅,

$$\mathcal{A}_{\text{rad}}(p_i; k) = \frac{\kappa}{2} \sum_i \frac{p_i^\mu p_i^\nu \varepsilon_{\mu\nu}(k)}{p_i \cdot k} \mathcal{A}(p_i), \tag{2}$$

is obtained from its gauge theory correspondent,

可由其规范理论对应形式得到,

$$\mathcal{A}_{\text{rad}}(p_i; k) = -g_s \sum_i t_i^a \frac{p_i \cdot \varepsilon^a(k)}{p_i \cdot k} \mathcal{A}(p_i), \tag{3}$$

by simply replacing the gauge charge (generator) by the gravitational charge, momentum, $t_i^a \to p_i^\nu$ and adjusting the coupling g_s and polarization vector $\varepsilon_\mu^a(k)$ to the gravitational coupling, $\kappa = \sqrt{32\pi G_N}$, and polarization tensor, $\varepsilon_{\mu\nu}(k)$. Since eikonalized propagators $1/(p_i \cdot k)$ are closely related to semi-infinite Wilson line operators, we expect soft graviton Wilson lines to play a similar role for soft graviton physics [33, 43] as they do in gauge theories. Equations (2) and (3) represent the leading terms in the so-called soft theorems, to which we shall return in a later section of this chapter.

通过简单地将规范荷 (生成元) 替换为引力荷、动量 $t_i^a \to p_i^\nu$ ，并将耦合 g_s 和极化矢量 $\varepsilon_\mu^a(k)$ 调整为引力耦合 $\kappa = \sqrt{32\pi G_N}$ 和极化张量 $\varepsilon_{\mu\nu}(k)$ 。由于程函化传播子 $1/(p_i \cdot k)$ 与半无限威尔逊线算符密切相关，我们预计软引力子威尔逊线在软引力子物理中 [33, 43] 会起到与它们在规范理论中类似的作用。方程 (2) 和 (3) 代表了所谓软定理中的主导项，我们将在本章的后续部分再讨论这些内容。

The collinear limit of graviton amplitudes is, however, very different from the one of gauge amplitudes. In fact, in gravity, collinear enhancements and singularities are absent altogether. As a consequence, even if the gravitational coupling was not minuscule, i.e., near Planckian scattering energies, energetic particles do not produce gravitational jets, which in QCD constitute the most visible footprints of the non-abelian charges of the quarks and gluons. The absence of collinear singularities for graviton emission was first shown in [41] in the simultaneous eikonal limit. Weinberg also noted that it would be rather troublesome, if this was not the case, since it would prevent the existence of massless particles with gravitational charges, that is, any non-vanishing four-momentum. However, while massless particles with gauge charges do not exist in Nature, and hence there is no conflict with the existence of collinear singularities in gauge theories, there are massless particles which gravitate, such as the photons and the gravitons themselves.

但引力子振幅的共线极限与规范振幅的共线极限截然不同。事实上，在引力中，共线增强和奇点完全不存在。因此，即使引力耦合不是极小，也就是在接近普朗克尺度的散射能量下，高能粒子也不会产生引力喷注，而在量子色动力学中，喷注是夸克和胶子非阿贝尔荷最明显的特征。引力辐射不存在共线奇点这一结论最早由文献 [41] 在同时程函极限下证明。温伯格也指出，如果情况并非如此，会带来相当大的麻烦，因为这会让带引力荷的无质量粒子也就是任何四动量非零的粒子无法存在。然而，自然界不存在带规范荷的无质量粒子，因此规范理论中共线奇点的存在并不矛盾，但存在会产生引力的无质量粒子，例如光子和引力子本身。

There is a simple qualitative explanation for the absence of collinear graviton singularities based on the classical radiation pattern [12]. When an energetic particle with virtuality much less than its three-momentum squared $\mathbf{p}^2$ emits a graviton with momentum $\mathbf{k}$ with small angle θ between $\mathbf{p}$ and $\mathbf{k}$, the near mass-shell singularity of the emitting particle propagator $1/(|\mathbf{p}|\,|\mathbf{k}|\,(1-\cos\theta))$ yields a factor θ^{-2} for the splitting amplitude. Quantizing the radiation field in the spherical basis with single-particle states $|\mathbf{k}jm;\lambda\rangle$, where λ denotes helicity (± 2 for gravitons and ± 1 for gauge bosons) and jm the angular momentum quantum numbers with respect to the quantization axis $\mathbf{p}$, this implies that the emitted graviton must be in a state $|\mathbf{k}j0;\lambda\rangle$, where $m=0$ due to angular momentum and helicity conservation. The | angular dependence of this state is given by the spin-weighted spherical harmonic or Wigner function $D^{j}_{\pm\lambda,0}(\mathbf{k}) \propto \sin^{|\lambda|}\frac{\theta}{2}$, which tends to zero as $\theta^{|\lambda|}$ in the $\theta \to 0$ limit. Thus, the splitting amplitude has no singularity in the collinear limit for graviton emission ($\lambda = \pm 2$) in contrast to the case of gauge bosons.

基于经典辐射图样，共线引力子奇点不存在可以得到一个简单的定性解释 [12]。当一个虚度远小于其三动量平方 $\mathbf{p}^2$ 的高能粒子发射一个动量为 $\mathbf{k}$ 的引力子，且 $\mathbf{p}$ 与 $\mathbf{k}$ 之间的夹角 θ 很小时，发射粒子传播子 $1/(|\mathbf{p}|\,|\mathbf{k}|\,(1-\cos\theta))$ 的近质壳奇点会给劈裂振幅带来一个因子 θ^{-2} 。将辐射场在球坐标基下量子化，得到单粒子态 $|\mathbf{k}jm;\lambda\rangle$ ，其中 λ 表示螺旋度 (引力子为 ± 2，规范玻色子为 ± 1)，jm 是相对于量子化轴 $\mathbf{p}$ 的角动量量子数，这表明出射引力子必须处于某个态 $|\mathbf{k}j0;\lambda\rangle$, where $m=0$ due to angular momentum and helicity conservation. The | ，该态的角依赖由带自旋球谐函数或维格纳函数 $D^{j}_{\pm\lambda,0}(\mathbf{k}) \propto \sin^{|\lambda|}\frac{\theta}{2}$ 给出，在 $\theta \to 0$ 极限下，该函数随 $\theta^{|\lambda|}$ 趋近于零。因此，与规范玻色子的情况不同，引力子发射 ($\lambda = \pm 2$) 的劈裂振幅在共线极限下不存在奇点。

The above argument refers to the physical polarization states of the graviton and thus does not cover the properties of individual Feynman amplitudes in general, in particular in covariant gauges, which do have collinear divergences. The formal demonstration of the absence of collinear divergences without the restriction to the eikonal limit adopted in [41] has been presented only relatively recently [1] with diagrammatic factorization methods. This fact is made evident in the construction of the soft-collinear effective Lagrangian for gravity (”soft-collinear gravity”) [12]: the leading effective Lagrangian describing collinear graviton self-interactions and their interactions with matter is a free theory. This motivates the investigation of collinear gravitational physics at sub-leading order in the collinear expansion, where it is nontrivial, and naturally leads to the systematic construction of soft-collinear effective gravity beyond the leading power in both, the collinear and soft limits [11].

上述论证针对的是引力子的物理极化态，因此没有涵盖一般情况下单个费曼振幅的性质，尤其是协变规范下的情况——协变规范中确实存在共线发散。不局限于文献 [41] 采用的 eikonal 极限，对共线发散不存在的形式证明直到近年才通过图因式分解方法给出 [1]。这一点在引力的软共线有效拉格朗日量 (即“软共线引力”) 的构造中十分明显 [12]: 描述共线引力子自相互作用及其与物质相互作用的领头阶有效拉格朗日量是一个自由理论。这启发我们研究共线展开中次领头阶的共线引力物理，该阶的物理非平庸，也自然引出在共线和软极限下，对超出领头幂次的软共线有效引力的系统构造 [11]。

The present chapter starts with a review of basic ideas and methods for soft-collinear Lagrangians, assuming no prior familiarity with the subject. We then provide a technically light-weight discussion of soft-collinear gravity, focusing on the exposition of the principles of the construction, the structure, and emergent symmetries of the result at the expense of many technical details, for which we refer to [11]. By definition, soft-collinear gravity builds an extension of the so-called soft theorems to all orders in the loop and soft expansion. It is nevertheless of interest to rederive them from the effective Lagrangian [10]. In the last section of this chapter, we briefly cover how this provides an understanding of why in gravity the soft theorem extends to the next-to-next-to-soft order (but does not in gauge theory) and how the form of the universal terms is related to the (emergent) soft gauge symmetries of the effective Lagrangian. The chapter concludes with a discussion of loop corrections to the soft theorem.

本章首先回顾软共线拉格朗日量的基本思想与方法，不需要读者预先熟悉该领域。随后我们对软共线引力给出技术上更简洁的讨论，侧重讲解构造原理、结果的结构与涌现对称性，省略了大量技术细节，相关细节可参见文献 [11]。根据定义，软共线引力将所谓软定理拓展到圈展开和软展开的所有阶。尽管如此，从有效拉格朗日量重新推导软定理仍然很有意义 [10]。在本章最后一节，我们简要介绍这一构造如何帮助我们理解: 为何引力中软定理可以拓展到次次软阶而规范理论中不行，以及普适项的形式如何与有效拉格朗日量的 (涌现) 软规范对称性相联系。本章最后讨论了软定理的圈修正。

Basic Ideas and Concepts

基本思想与概念

The following section sets up the notation and introduces a number of concepts that arise in effective field theory (EFT) and SCET in particular.

本节将确立符号约定，介绍有效场论 (EFT) 尤其是软共线有效理论 (SCET) 中出现的若干核心概念。

Perturbative Gravity

微扰引力

The full theory, from which SCET gravity is constructed, is the Einstein-Hilbert theory with action

构造 SCET 引力所依据的完整理论是带有如下作用量的爱因斯坦-希尔伯特理论

$$S_{\mathrm{EH}} = -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}R, \tag{4}$$

coupled to matter, here a minimally coupled scalar field φ in the curved space-time with metric tensor $g_{\mu\nu}$. The matter part is described by the action

它与物质耦合，此处为弯曲时空中最小耦合的标量场 φ ，弯曲时空带有度量张量 $g_{\mu\nu}$ 。物质部分由如下作用量描述

$$S_{\varphi} = \int d^4x\sqrt{-g}\frac{1}{2}g^{\mu\upsilon}\partial_{\mu}\varphi\partial_{\upsilon}\varphi, \tag{5}$$

where g denotes the metric determinant. [1] At this point, there is an important notion to clarify: the Einstein-Hilbert action is not renormalizable in the strict sense. Instead, it should be treated as the first term in a low-energy EFT of gravity. This idea was pioneered in [26] and is also well explained in [27]. If gravitational loops are included, it is necessary to introduce additional terms to render the theory finite. These higher-order terms correspond to a derivative expansion, and they can be expressed as products of Riemann tensors. Schematically, the full action then takes the form

其中 g 表示度量行列式。[1] 在此处需要澄清一个重要概念: 爱因斯坦-希尔伯特作用量在严格意义上是不可重整的，它应当被视作引力低能有效场论的首项。这一思想由文献 [26] 开创，在文献 [27] 中也有详尽解释。如果纳入引力圈图修正，就必须引入额外项使理论有限，这些高阶项对应导数展开，可以表示为黎曼张量的乘积。概要而言，完整作用量形式如下

$$S_{\text{grav, EFT}} = -\int d^4x\sqrt{-g}\left(\Lambda + \frac{2}{\kappa^2}R - c_1R^2 - c_2R_{\mu\nu}R^{\mu\nu} + \dots\right), \tag{7}$$

where Λ is the cosmological constant and $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ is the Ricci tensor. This subtlety is important if one wants to consider higher-loop orders in gravity, though this is not very relevant for the following discussion. While the detailed form of the soft-collinear effective theory is determined by the full theory from which it is derived, the construction of the effective theory does not depend on the precise loop order that is considered. All these higher-order terms respect the same gauge symmetry that is already present in the leading term, the diffeomorphism invariance, and this symmetry forms the guiding principle of the SCET construction.

其中 Λ 是宇宙学常数，$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ 是里奇张量。对于要研究引力高阶圈图修正的情况，这一细微之处十分重要，不过对于下文的讨论而言它并不关键。虽然软共线有效理论的具体形式由它所源出的完整理论决定，但有效理论的构造并不依赖所考虑的精确圈阶。所有这些高阶项都满足首项已具备的相同规范对称性——微分同胚不变性，该对称性正是 SCET 构造的指导原则。

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[1] In the following, the convention

[1] 下文采用的约定

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha\Gamma^{\mu}_{\beta\upsilon} - \partial_\beta\Gamma^{\mu}_{\alpha\upsilon} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\beta\upsilon} - \Gamma^{\mu}_{\beta\lambda}\Gamma^{\lambda}_{\alpha\upsilon} \tag{6}$$

for the Riemann tensor and the metric signature $(+,-,-,-)$ are employed.

是黎曼张量与度量符号 $(+,-,-,-)$。

The diffeomorphisms can be arranged in the form of local translations

微分同胚可以整理为局域平移的形式

$$x^\mu \to x'^\mu(x) = x^\mu + \varepsilon^\mu(x), \tag{8}$$

where $\varepsilon^\mu(x)$ is some (not necessarily small) vector field. Under such a transformation, scalar fields behave as

其中 $\varepsilon^\mu(x)$ 是任意 (不一定是小量) 矢量场。在这类变换下，标量场的变换行为为

$$\varphi(x) \to \varphi'(x') \stackrel{!}{=} \varphi(x). \tag{9}$$

The transformed scalar field $\varphi'(x')$ can be expressed as

变换后的标量场 $\varphi'(x')$ 可以写为

$$\varphi'(x + \varepsilon(x)) \equiv T_\varepsilon \varphi'(x), \tag{10}$$

where the translation operator T_ε is defined as

其中平移算符 T_ε 定义为

$$T_\varepsilon f(x) = f(x) + \varepsilon^\alpha(x)\partial_\alpha f(x) + \frac{1}{2}\varepsilon^\alpha(x)\varepsilon^\beta(x)\partial_\alpha\partial_\beta f(x) + \mathcal{O}(\varepsilon^3). \tag{11}$$

In the following, this active point of view is adopted. That is, the local translations correspond to purely internal transformations, acting on the field space, instead of actually transforming the coordinates. In practice, this means that the dynamical fields $\varphi(x) \to \varphi'(x)$ transform, but never the coordinates x themselves. A scalar field then transforms as

下文采用这种主动观点: 即局域平移对应纯内部变换，作用在场空间上，而非真正变换坐标。实际应用中，这意味着只有动力学场 $\varphi(x) \to \varphi'(x)$ 发生变换，坐标 x 本身不发生变换。因此标量场的变换为

$$\varphi(x) \to \varphi'(x) = [U(x)\varphi(x)], \tag{12}$$

where due to (10) $U(x)$ is the inverse translation, $U(x) = T_\varepsilon^{-1}$. [2] The metric field transforms as

其中由式 (10) 可知 $U(x)$ 是逆平移，$U(x) = T_\varepsilon^{-1}$. [2] 度量场的变换为

$$g_{\mu\nu}(x) \to \left[U(x)\left(U_\mu{}^\alpha(x)\, U_\nu{}^\beta(x)\, g_{\alpha\beta}(x)\right)\right], \tag{13}$$

with Jacobi matrices

其中雅可比矩阵为

$$U_\alpha^\mu(x) = \frac{\partial x'^\mu}{\partial x^\alpha}(x),\ U_\mu^\alpha(x) = \frac{\partial x^\alpha}{\partial x'^\mu}(x). \tag{14}$$

[2] The square-bracket notation emphasizes that $U(x)$ is a derivative operator that acts only on $\varphi(x)$.

[2] 方括号记号强调 $U(x)$ 是仅作用于 $\varphi(x)$ 的导数算符。

It is convenient to adopt this active point of view for two reasons: First, it emphasizes the formal similarity to gauge theories. Note that the transformation (12) looks formally the same as the transformation of a matter field $\phi^a(x)$ with respect to a non-abelian gauge symmetry,

采用主动观点有两个便捷之处: 首先，它突出了该理论与规范理论的形式相似性。注意变换 (12) 在形式上和非阿贝尔规范对称性下物质场 $\phi^a(x)$ 的变换完全一致,

$$\phi^a(x) \to U^{ab}(x)\phi^b(x), \tag{15}$$

which then exposes the similarities and differences between these transformations in gravity and gauge theory. Second, in this active point of view, one never has to worry about transformations of the coordinates, the integral measure d^4x as well as derivatives $\frac{\partial}{\partial x^\mu}$, and only needs to keep track of the transformation of the dynamic fields. Note that this point of view does not change the form of invariant objects. In the passive point of view, $\int d^4x\sqrt{-g}\mathcal{L}$ is diffeomorphism-invariant, since the invariant measure $d^4x\sqrt{-g}$ appears in combination with a scalar quantity $\mathcal{L}$. In the active point of view, the relevant object is the scalar density $\sqrt{-g}\mathcal{L}$, which is gauge-invariant up to total derivatives, rendering the integral manifestly invariant. This provides a guiding principle for the required manipulations, such as the construction of gauge-invariant or gauge-covariant fields.

这进而揭示了引力与规范理论中这些变换的异同。其次，在这种主动观点下，我们无需担心坐标、积分测度 d^4x 以及导数 $\frac{\partial}{\partial x^\mu}$ 的变换，只需要跟踪动力学场的变换即可。请注意，该观点不会改变不变量对象的形式。在被动观点下，$\int d^4x\sqrt{-g}\mathcal{L}$ 是微分同胚不变的，因为不变测度 $d^4x\sqrt{-g}$ 与标量 $\mathcal{L}$ 结合出现。在主动观点下，相关对象是标量密度 $\sqrt{-g}\mathcal{L}$ ，它在全导数范围内是规范不变的，因此积分显然是不变的。这为所需的操作 (例如规范不变场或规范协变场的构造) 提供了指导原则。

To construct perturbative gravity, one assumes small fluctuations of the metric field and performs a weak-field expansion of

为了构造微扰引力，我们假设度规场存在小涨落，并对其进行弱场展开

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \tag{16}$$

in $h_{\mu\nu}(x)$ around Minkowski space with metric $\eta_{\mu\nu}$. The action then turns into an infinite series in $h_{\mu\nu}$, resp. κ :

在 $h_{\mu\nu}(x)$ 中围绕度规为 $\eta_{\mu\nu}$ 的闵氏空间展开。作用量随后变为 $h_{\mu\nu}$ (即 κ) 的无穷级数:

$$S = \sum_{k=0}^{\infty} \kappa^k S^{(k)} \tag{17}$$

where the precise form of $S^{(k)}$ at higher orders depends on which terms one considers to be part of the ”full theory,” i.e., if one only considers Einstein-Hilbert (4) or also takes into account higher-order Riemann terms as in (7).

其中高阶项 $S^{(k)}$ 的精确形式取决于我们将哪些项视为“完整理论”的一部分，即仅考虑爱因斯坦-希尔伯特项 (4)，还是也纳入 (7) 中的高阶黎曼项。

In this weak-field expansion, the residual gauge transformations correspond to the translations [3]

在该弱场展开中，剩余规范变换对应平移 [3]

$$x^{\mu} \to x^{\mu} + \kappa\varepsilon^{\mu}(x)\,. \tag{18}$$

The action of $U(x)$ is given by

$U(x)$ 的作用由下式给出

$$[U(x)\varphi(x)] = \varphi(x) - \kappa\varepsilon^{\alpha}(x)\,\partial_{\alpha}\varphi(x) \tag{19}$$

$$+\frac{\kappa^2}{2}\varepsilon^{\alpha}(x)\,\varepsilon^{\beta}(x)\,\partial_{\alpha}\partial_{\beta}\varphi(x) + \kappa^2\varepsilon^{\alpha}(x)\,\partial_{\alpha}\varepsilon^{\beta}(x)\,\partial_{\beta}\varphi(x) + \mathcal{O}\left(\varepsilon^3\right).$$

To first order, the transformations (12) and (13) reproduce the well-known results

一阶下，变换 (12) 和 (13) 给出了熟知的结果

$$h_{\mu\upsilon} \to h_{\mu\upsilon} - \partial_{\mu}\varepsilon_{\upsilon} - \partial_{\upsilon}\varepsilon_{\mu} + \mathcal{O}\left(\varepsilon^2, \varepsilon h\right),$$

(20)

$$\varphi \to \varphi - \kappa\varepsilon^{\alpha}\partial_{\alpha}\varphi + \mathcal{O}\left(\varepsilon^2\right).$$

---

[3] We extract a factor of κ from ε^{μ} , so that $h_{\mu\nu}$ has the linear gauge transformation (20) that does not contain κ explicitly.

[3] 我们从 ε^{μ} 中提取出因子 κ ，因此 $h_{\mu\nu}$ 满足线性规范变换 (20)，该变换不显含 κ 。

An immediate consequence of these truncated diffeomorphisms is that objects which are homogeneous in h , that is, are monomials in h , cannot be gauge-invariant at the same time. Gauge-invariant objects, such as the Riemann tensor, are represented as a series order by order in h or κ . When working with a theory expanded in h , sub-leading terms, that is, higher-order terms in h , appear in precise combinations to yield a gauge-invariant theory. This is a generic feature of nonlinearly realized symmetries.

这些截断微分同胚的一个直接结论是: 对 h 齐次的对象 (即 h 中的单项式) 无法同时满足规范不变。黎曼张量这类规范不变对象可表示为 h 或 κ 下按阶展开的级数。当我们使用按 h 展开的理论时，次领头项 (即 h 中的高阶项) 会以精确组合的形式出现，得到规范不变的理论。这是非线性实现对称性的一般特征。

Basic Concepts of SCET

SCET 基本概念

SCET is the theory describing the (self-)interactions of soft and collinear particles [2, 3, 5, 6] . It is one of the more complicated effective theories in modern particle physics and is used to great success in high-energy collider physics. Despite its technical nature, one can intuitively understand the construction with only a few key concepts. It is this intuition that is paramount in constructing the gravitational analogue. This section serves as an introduction into the underlying concepts of SCET. While the section aims to be self-contained, it will focus on exposition rather than derivations. Detailed discussions and computations can be found in [5, 7, 10, 11] .

软-共线有效场论 (SCET) 是描述软粒子和共线粒子的 (自) 相互作用的理论 [2, 3, 5, 6] 。它是现代粒子物理中较为复杂的有效场论之一，目前已在高能对撞机物理中取得极大成功。尽管它具备较强的技术属性，仅需几个核心概念就能直观理解其构造，而这种直观理解对构建引力类比模型至关重要。本节将介绍 SCET 的基础概念，本节虽力求内容自洽，但重点放在原理阐释而非推导上，详细的讨论与计算可参见 [5, 7, 10, 11] 。

Kinematics and Power Counting

运动学与幂次计数

The general kinematics underlying SCET consist of an energetic scattering, characterized by its large energy scale Q of some hard process, which creates a number of energetic particles, as well as some low-energy, soft radiation. These energetic particles are called ”collinear” and develop into jets by collinear radiation. The jets are assumed to be well separated in angle from each other. The situation is depicted in Fig. 2.

SCET 的基础一般运动学对应硬过程中特征为大能量标度 Q 的高能散射，该过程会产生大量高能粒子，以及一些低能软辐射。这些高能粒子被称为“共线粒子”，并通过共线辐射演化形成喷注。假设各喷注在角度上彼此完全分离，该情形如图 2 所示。

To be precise, the energetic particles of jet i are taken to be ultrarelativistic (light-like) and characterized by the light-like direction n_{i-}^{μ} of the jet with respect to which they have small transverse momentum. Corresponding to each n_{i-}^{μ} , there is a n_{i+}^{μ} such that $n_{i+} \cdot n_{i-} = 2, n_{i\pm}^2 = 0$. These two reference vectors, as well as the two remaining transverse directions, form a basis, with metric tensor

准确来说，喷注 i 的高能粒子为极端相对论性 (类光) 粒子，以喷注的类光方向 n_{i-}^{μ} 为特征，粒子相对于该方向仅带有小横动量。对应每个 n_{i-}^{μ} ，都存在一个 n_{i+}^{μ} 满足 $n_{i+}\cdot n_{i-}=2, n_{i\pm}^2=0$ 。这两个参考矢量与剩下的两个横向方向构成一组基，其度规张量为

$$\eta^{\mu\nu}=\frac{1}{2}n_{i+}^{\mu}n_{i-}^{\nu}+\frac{1}{2}n_{i+}^{\nu}n_{i-}^{\mu}+\eta_{\perp i}^{\mu\nu}. \tag{21}$$

One can decompose a collinear momentum p^{μ} as

我们可以将共线动量 p^{μ} 分解为

$$p^{\mu}=n_{i+}p\frac{n_{i-}^{\mu}}{2}+p_{\perp i}^{\mu}+n_{i-}p\frac{n_{i+}^{\mu}}{2}, \tag{22}$$

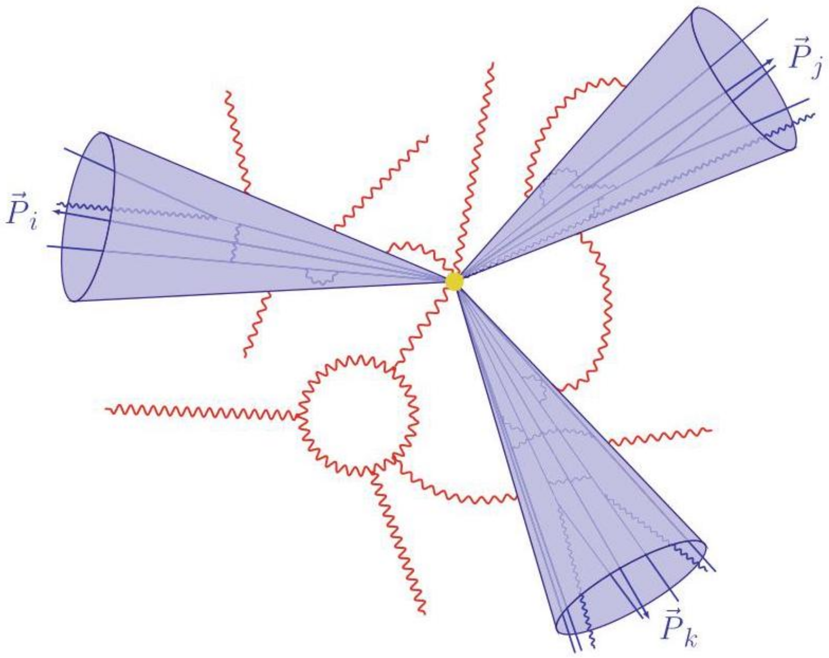

Fig. 2 The generic kinematic situation of a scattering process described by SCET. Blue lines denote collinear particles forming jets, emanating from a point-like hard interaction. Red lines represent soft modes connecting jets

图 2 SCET 描述的散射过程的一般运动学情形。蓝线表示从点相互作用硬作用点出射、形成喷注的共线粒子，红线表示连接各喷注的软模。

where the subscript $\perp i$ denotes the transverse components with respect to $n_{i+}^{\mu}, n_{i-}^{\mu}$. Introducing the power counting parameter $\lambda\sim p_{\perp i}/(n_{i+}p)\ll 1$, the components of the collinear momentum p scale as

其中下标 $\perp i$ 表示相对于 $n_{i+}^{\mu}, n_{i-}^{\mu}$ 的横向分量。引入幂次计数参数 $\lambda\sim p_{\perp i}/(n_{i+}p)\ll 1$ 后，共线动量 p 各分量的标度为

$$(n_{i+}p, p_{\perp i}, n_{i-}p) \sim \left(1, \lambda, \lambda^2\right) Q \tag{23}$$

and have virtuality $p^2 = \lambda^2 Q^2$. It is a convention to set $Q = 1$, which is adopted in the following. The condition $\lambda \ll 1$ means that the momentum flows in the direction n_{i-}^{μ} and only small fractions are deposited in the transverse directions. For multiple particles in the same collinear sector, it means that these are separated by a small angle; thus, they constitute a jet. Besides the collinear modes, also soft modes can be present, which can interact with collinear modes without changing their collinear nature. This implies isotropic momentum scaling $k^{\mu} \sim \lambda^2$ and virtuality $k^2 \sim \lambda^4$, parametrically smaller than collinear virtuality. Soft modes can be exchanged between the energetic particles in different directions, as shown in Fig. 2. This kinematic situation, consisting only of collinear and soft modes of different virtuality, is usually denoted as SCET $_{\mathrm{I}}$. In this chapter, we refer exclusively to this situation.

其虚度为 $p^2 = \lambda^2 Q^2$ 。约定取 $Q = 1$ ，下文也沿用该约定。条件 $\lambda \ll 1$ 意味着动量沿 n_{i-}^{μ} 方向流动，仅极小份额的动量分布在横向方向。对于同一共线区中的多个粒子，该条件意味着粒子间仅以小角度分离，因此它们共同构成一个喷注。除共线模外，还可以存在软模，软模可与共线模相互作用，且不改变共线模的共线性质。这说明软模满足各向同性动量标度 $k^{\mu} \sim \lambda^2$ ，虚度为 $k^2 \sim \lambda^4$ ，其参数大小小于共线虚度。如图 2 所示，软模可以在不同方向的高能粒子之间交换。这种仅包含不同虚度的共线模与软模的运动学情形通常记为 SCET $_{\mathrm{I}}$ ，本章仅讨论该情形。

Field Content

场内容

The goal is to construct an EFT that provides a systematic expansion for the full theory scattering amplitudes in the soft and collinear limits. The construction differs considerably from traditional EFTs, such as the Fermi theory of weak decays. In these theories, one is interested in ”light physics” and integrates out the ”heavy fields,” with masses above some scale Λ . This gives rise to an effective Lagrangian wherein only the light fields are dynamical, which now contains (local) higher-dimensional operators, that describe the short-distance physics. The dimension of these operators serves as power counting parameter for the expansion in $1/\Lambda$.

我们的目标是构建一个有效场论 (EFT)，对软极限和共线极限下完整理论的散射振幅进行系统展开。该构建过程与传统有效场论 (例如弱衰变的费米理论) 差异很大。在这些传统理论中，研究者关注“轻物理”，并积分掉质量高于某个标度 Λ 的“重场”。由此得到的有效拉氏量中只有轻场是动力学的，且拉氏量包含描述短程物理的 (定域) 高维算符。这些算符的维度可作为 $1/\Lambda$ 展开中的幂次计数参数。

In SCET, however, the soft and collinear regions of all relevant full theory particles can contribute to scattering amplitudes. Instead of integrating out heavy fields, one integrates out certain fluctuations of the fields or, equivalently, certain regions of momenta. Specifically, one integrates out the hard regions of momenta while keeping collinear and soft ones as the degrees of freedom in the EFT. In order to achieve this systematically at the Lagrangian level, one needs to split the full theory fields ϕ_J into hard, soft, and collinear modes $\phi_{J,h}, \phi_{J,s}$, and ϕ_{J,c_i} .

然而在软共线有效理论 (SCET) 中，完整理论所有相关粒子的软区和共线区都可以对散射振幅有贡献。该理论不是积分掉重场，而是积分掉场的特定涨落，或者等价地说，积分掉特定的动量区域。具体来说，我们积分掉硬动量区域，将共线动量和软动量保留为有效场论的自由度。为了在拉氏量层面系统地实现这一步，我们需要将完整理论场 ϕ_J 拆分为硬模、软模和共线模 $\phi_{J,h}, \phi_{J,s}$ 和 ϕ_{J,c_i}。

To construct a systematic expansion, these modes must be homogeneous in λ, that is, they must scale with a unique power of λ, in order to label them as hard, collinear, and soft. For this reason, the EFT uses the different fields ϕ_{J,c_i} and $\phi_{J,s}$, which specifically describe the fluctuations of the original field in the respective kinematic regions. For SCET, the relevant modes are soft and collinear ones, as depicted in Fig. 2. In addition, each term in the Lagrangian must be fully expanded to also be homogeneous, in the sense that it does not contain any further sub-leading terms.

为了构建系统展开，这些模必须对 λ 是齐次的，即它们都对应 λ 的唯一幂次，这样才能将它们标记为硬模、共线模和软模。因此，该有效场论使用不同的场 ϕ_{J,c_i} 和 $\phi_{J,s}$，分别描述原场在对应运动学区域的涨落。软共线有效理论中，相关模为软模和共线模，如图 2 所示。此外，拉氏量中的每一项都必须完全展开，使其也是齐次的，即不包含任何额外的次领头项。

Since the components n_{i+p} of collinear fluctuations are of the order of the hard scale, one cannot expand in $n_{i+}p/Q$, and the effective Lagrangian cannot be local. To see this, note that any operator can feature an arbitrary number of large derivatives $n_{i+}\partial$, which all scale as $n_{i+}\partial \sim 1$ and are thus in principle present at any order in λ. However, this tower of derivatives can be traded for the non-locality in the n_{i+}^{μ} direction. This is possible simply by rewriting the derivatives as a translation, using

由于共线涨落的分量 n_{i+p} 是硬标度量级，我们无法对 $n_{i+}p/Q$ 做展开，因此有效拉氏量不能是定域的。不难发现，任意算符都可以包含任意数量的大导数 $n_{i+}\partial$，这些导数都标度为 $n_{i+}\partial \sim 1$，因此原则上它们在 λ 的任意阶都存在。不过这一系列导数可以转化为 n_{i+}^{μ} 方向的非定域性，只需利用平移关系改写导数即可:

$$\phi_{J,c_i}(x + tn_{i+}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (n_{i+}\partial)^k \phi_{J,c_i}(x). \tag{24}$$

Thus, instead of keeping track of large derivatives $n_{i+}\partial$, one allows for non-localities in the n_{i+}^{μ} direction for collinear objects. When several energetic particles scatter at large angles, the theory is therefore non-local along the directions n_{i+}^{μ}, i.e., orthogonal to the classical trajectories of the collinear particles. It is, however, local along these trajectories, resp. the direction n_{i-}^{μ} of their momentum.

因此，我们不需要追踪大导数 $n_{i+}\partial$，只需要允许共线对象在 n_{i+}^{μ} 方向存在非定域性。当多个高能粒子发生大角度散射时，理论会在垂直于共线粒子经典轨迹的方向 n_{i+}^{μ} 上是非定域的，但沿着这些轨迹即动量方向 n_{i-}^{μ} 仍然是定域的。

In conventional effective theories, the importance of a local operator is tied to its mass dimension. In SCET, however, the power counting is not related to the mass dimension. In particular, different components of a field can acquire different scaling in λ. To determine the power counting of these fields, one considers the two-point function

在传统有效理论中，定域算符的重要性由其质量维度决定。但在软共线有效理论中，幂次计数与质量维度无关。具体来说，同一个场的不同分量在 λ 下可以有不同的标度。为了确定这些场的幂次计数，我们考虑两点函数:

$$\langle 0|T\varphi(x)\varphi(0)|0\rangle = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4(\lambda^8)} e^{-ipx} \underbrace{\frac{i}{p^2+i0}}_{\lambda^{-2}(\lambda^{-4})} \sim \lambda^2(\lambda^4), \tag{25}$$

where the power counting of collinear (soft) momenta is given explicitly. One obtains

其中已经明确给出了共线 (软) 动量的幂次计数。我们得到

$$\varphi_{c_i}(x) \sim \lambda,\ \varphi_s(x) \sim \lambda^2 \tag{26}$$

for the collinear and soft scalar field, respectively. For the graviton field $h_{\mu\nu}(x)$, one first performs the weak-field expansion and fixes a general de Donder gauge with parameter b . Then, the two-point function reads

分别对应共线标量场和软标量场。对于引力子场 $h_{\mu\nu}(x)$ ，我们首先做弱场展开，固定带参数 b 的一般德东规范。此时两点函数为

$$\langle 0|Th_{\mu\nu}(x)h_{\alpha\beta}(0)|0\rangle = i\kappa^2 \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2+i0}\left(P_{\mu\nu,\alpha\beta}+\frac{1-b}{b}S_{\mu\nu,\alpha\beta}\right),$$

(27)

where

其中

$$P_{\mu\upsilon,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\upsilon\beta}+\eta_{\mu\beta}\eta_{\upsilon\alpha}-\eta_{\mu\upsilon}\eta_{\alpha\beta}\right),$$

$$S_{\mu\upsilon,\alpha\beta} = \frac{1}{2p^2}\left(\eta_{\mu\alpha}p_\upsilon p_\beta+\eta_{\mu\beta}p_\upsilon p_\alpha+p_\mu p_\alpha\eta_{\upsilon\beta}+p_\mu p_\beta\eta_{\upsilon\alpha}\right). \tag{28}$$

Inserting the collinear momentum scaling, one obtains $P_{\mu\nu,\alpha\beta} \sim 1$ if it is nonvanishing, since it does not depend on momenta. The other combination, $S_{\mu\nu,\alpha\beta}$, has non-trivial λ -scaling. For example, for the $\perp+$ and $\perp\perp$ modes, [4] one obtains

代入共线动量标度后，由于 $P_{\mu\nu,\alpha\beta} \sim 1$ 不依赖动量，若它非零则其标度为常数。另一组合 $S_{\mu\nu,\alpha\beta}$ 则具有非平凡的 λ 标度。例如，对于 $\perp+$ 模和 $\perp\perp$ 模 [4] ，我们得到

$$S_{\perp+,\perp+} \sim \frac{1}{\lambda^2},\ P_{\perp+,\perp+} = 0,$$

$$S_{\perp\perp,\perp\perp} \sim 1,\ P_{\perp\perp,\perp\perp} \sim 1. \tag{29}$$

Then, the scaling of the components of the collinear graviton field is easily determined from (27) to be [12]

随后，根据式 (27) 可以很容易得到共线引力子场各分量的标度为 [12]:

$$h_{++} \sim \lambda^{-1},\ h_{+\perp} \sim 1,\ h_{+-} \sim \lambda,$$

$$h_{--} \sim \lambda^{3},\ h_{-\perp} \sim \lambda^{2},\ h_{\perp\perp} \sim \lambda, \tag{30}$$

which implies $h^{\mu}{}_{\mu} \equiv h \sim \lambda$ for the trace. Note that the theory contains the $\mathcal{O}(1)$ field component $h_{\mu_\perp +} \sim 1$ and even a power-enhanced component $h_{++} \sim \lambda^{-1}$. This is problematic for the λ -expansion in the effective theory and must be addressed in the construction of the theory. For now, observe that coupling $h_{\mu\nu}$ to a vector V^{μ} of the same collinearity yields

这对迹而言隐含了 $h^{\mu}{}_{\mu} \equiv h \sim \lambda$。注意该理论包含 $\mathcal{O}(1)$ 场分量 $h_{\mu_\perp +} \sim 1$， 甚至还存在一个幂次增强分量 $h_{++} \sim \lambda^{-1}$ 。这对有效理论中的 λ 展开而言存在问题，必须在理论构建过程中加以处理。现在可以观察到，将耦合 $h_{\mu\nu}$ 与同共线性的矢量 V^{μ} 结合可得

$$h_{\mu\upsilon}V^{\upsilon} = \frac{1}{2}\left(h_{\mu+}V_{-} + h_{\mu-}V_{+}\right) + h_{\mu\upsilon_\perp}V^{\upsilon_\perp} \sim \lambda V_{\mu}, \tag{31}$$

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[4] For a tensor index $\mu, +, -$ means $T_{\pm} \equiv n^{\mu}_{\pm}T_{\mu}$, while $\perp$ stands for $T_{\mu_\perp}$. In the remainder of section "Basic Ideas and Concepts," we drop the collinear direction label i whenever referring to a single collinear sector.

[4] 对张量指标 $\mu, +, -$ 而言意味着 $T_{\pm} \equiv n^{\mu}_{\pm}T_{\mu}$ ，而 $\perp$ 代表 $T_{\mu_\perp}$ 。在“基本思想与概念”小节余下内容中，当提及单个共线扇区时，我们将省略共线方向标号 i

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so index contractions within the same collinear sector are suppressed by a power of λ . This argument does not hold for couplings between different sectors, where h_{++} could give rise to power enhancement. For the soft graviton, it is straightforward to derive the isotropic scaling $s_{\mu\upsilon} \sim \lambda^2$ from its propagator.

因此同一共线扇区内的指标缩并会被一个 λ 幂次压低。这个论证不适用于不同扇区间的耦合，不同扇区中 h_{++} 会引发幂次增强。对于软引力子，可以直接从其传播子推导出各向同性标度 $s_{\mu\upsilon} \sim \lambda^2$

Gauge Symmetry

规范对称性

Next, it is useful to discuss the gauge symmetry of SCET. In the full theory, there is only one gauge symmetry, diffeomorphism invariance. Each full theory field furnishes some representation of the diffeomorphism group and comes with its own gauge transformation. For the scalar field and the graviton, the linear transformation is given in (20). Next, one performs the mode split, that is, for each full theory field $h_{\text{full},\mu\nu}(x)$ and $\varphi(x)$, one obtains collinear modes $h_{\mu\nu}(x)$ and $\varphi_c(x)$ and soft modes $s_{\mu\nu}(x)$ and $\varphi_s(x)$. This has implications for the gauge symmetry in the EFT. Once the split is implemented, e.g., naively as $h_{\text{full},\mu\nu} = h_{\mu\nu} + s_{\mu\nu}$, the right-hand side has to transform like the full theory graviton on the left. This gives constraints on the allowed gauge symmetry, since the two fields on the right-hand side are modes with homogeneous power counting by construction. The soft field can never transform with a gauge parameter that contains collinear fluctuations, since this would turn the soft field into a collinear field. But such gauge parameters are allowed in the transformation of $h_{\mu\nu}$. The solution is to extend the gauge symmetry to two separate, collinear and soft, gauge symmetries such that the collinear fields take the role of fluctuations on top of the soft background $g_{s\mu\nu} = \eta_{\mu\nu} + \kappa s_{\mu\nu}$. The technical details will be presented in section "Soft-Collinear Gravity." The collinear gauge transformation then reads

接下来，讨论 SCET 的规范对称性是很有必要的。在完整理论中仅存在一种规范对称性，即微分同胚不变性。完整理论的每个场都对应微分同胚群的某种表示，且有自身的规范变换。对于标量场和引力子，线性变换由式 (20) 给出。接下来我们进行模式分解，也就是说，对完整理论的每个场 $h_{\text{full},\mu\nu}(x)$ 和 $\varphi(x)$ ，可以得到共线模式 $h_{\mu\nu}(x)$ 和 $\varphi_c(x)$ 以及软模式 $s_{\mu\nu}(x)$ 和 $\varphi_s(x)$ 。这会影响有效场论中的规范对称性。一旦完成分解 (例如，像 $h_{\text{full},\mu\nu} = h_{\mu\nu} + s_{\mu\nu}$ 那样的朴素分解)，等式右侧必须和左侧的完整理论引力子满足相同的变换规律。这会对允许的规范对称性给出约束，因为等式右侧的两个场按构造就是满足齐次幂计数的模式。软场永远不能用包含共线涨落的规范参数变换，因为这会将软场转变为共线场。但这类规范参数在 $h_{\mu\nu}$ 的变换中是允许的。解决方案是将规范对称性扩展为两个独立的规范对称性: 共线规范对称性与软规范对称性，使得共线场扮演软背景 $g_{s\mu\nu} = \eta_{\mu\nu} + \kappa s_{\mu\nu}$ 上的涨落。技术细节将在“软共线引力”一节给出。共线规范变换可写为

$$\kappa h_{\mu\nu} \to \left[U_c\left(U_{c\mu}{}^{\alpha} U_{c\nu}{}^{\beta}\left(g_{s\alpha\beta} + \kappa h_{\alpha\beta}\right)\right)\right] - g_{s\mu\nu},$$

$$g_{s\mu\nu} \to g_{s\mu\nu} \tag{32}$$

The intuition behind these transformations is clear: the fluctuation $h_{\mu\nu}$ comes with its own gauge symmetry, but the background $g_{s\mu\nu}$ is unaffected by this collinear transformation, as it must be. The fluctuation $h_{\mu\nu}$ transforms in the same way as the $h_{\mu\nu}$ of the weak-field expansion in the full theory, except that the rigid Minkowski background $\eta_{\mu\nu}$ is replaced by $g_{s\mu\nu}$ in (32). The soft background field is itself dynamic, with $s_{\mu\nu}$ transforming under the soft gauge symmetry $U_s(x)$ (to be discussed in section "Soft-Collinear Gravity"). Let us stress the main message: to consistently implement the split into soft and collinear modes, one treats the collinear fields as fluctuations on top of a soft background. [5]

这些变换背后的直观物理图像很清晰: 涨落 $h_{\mu\nu}$ 自带其规范对称性，但背景 $g_{s\mu\nu}$ 不受该共线变换的影响，这符合理应如此的要求。涨落 $h_{\mu\nu}$ 的变换方式与完整理论弱场展开中 $h_{\mu\nu}$ 的变换方式相同，只是 (32) 式中的刚性闵氏背景 $\eta_{\mu\nu}$ 被替换为 $g_{s\mu\nu}$ 。软背景场本身是动力学的，其中 $s_{\mu\nu}$ 在软规范对称性 $U_s(x)$ 下变换 (将在“软共线引力”一节讨论)。我们需要强调核心结论: 要一致地完成软模式和共线模式的分解，需要将共线场视为软背景之上的涨落。[5]

Light-Front Multipole Expansion

光前多极展开

Due to the power counting of their momenta (and thus coordinate arguments), products containing both soft and collinear fields are not homogeneous in λ . In Fourier space, one finds, for example,

由于软场和共线场的动量 (以及坐标自变量) 功率计数，同时包含二者的乘积对 λ 不是齐次的。例如在傅里叶空间中可以得到:

$$\varphi_c(x)\varphi_s(x) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k_s}{(2\pi)^4} e^{-i(p+k_s)\cdot x}\tilde{\varphi}_c(p)\tilde{\varphi}_s(k_s). \tag{33}$$

Here, the product in the exponent reads

此处，指数上的乘积为

$$(p+k_s)\cdot x = \frac{1}{2}\underbrace{(n_+p+n_+k_s)}_{1+\lambda^2}n_-x + \underbrace{(p_\perp + k_{s\perp})}_{\lambda+\lambda^2}\cdot x_\perp + \frac{1}{2}\underbrace{(n_-p+n_-k_s)}_{\lambda^2+\lambda^2}n_+x.$$

(34)

Only the combination

仅当组合

$$n_-p + n_-k_s \sim \lambda^2 \tag{35}$$

scales homogeneously as $\mathcal{O}(\lambda^2)$, while the $k_{s\perp}$ and n_+k_s components are suppressed with respect to $p_\perp$ and n_+p . The physical reason for this is that soft fields can only resolve large distances

对标度变化满足 $\mathcal{O}(\lambda^2)$ 下的齐次性，而 $k_{s\perp}$ 分量和 n_+k_s 分量相对 $p_\perp$ 和 n_+p 是压低的。其物理原因是软场仅能分辨大尺度距离

$$x_-^\mu \equiv n_+x\frac{n_-^\mu}{2} \tag{36}$$

whereas collinear fields fluctuate also over smaller distances. Therefore, one must expand the exponential in (33) as

而共线场还可以在更小的距离上涨落。因此，必须将 (33) 式中的指数展开为

$$e^{-i(p+k_s)\cdot x} = e^{-i\left(p+n_-k_s\frac{n_+}{2}\right)\cdot x}\left(1 - ik_{s\perp}\cdot x_\perp - \frac{i}{2}n_+k_sn_-x + \ldots\right), \tag{37}$$

keeping only the homogeneous soft momentum n_-k_s in the exponent. This expansion is equivalent to the light-front multipole expansion [5, 6]

指数中仅保留齐次软动量 $n_- k_s$ 。该展开等价于光前多极展开 [5, 6]

$$\varphi_s(x) = \varphi_s(x_-) + (x - x_-)^\mu \left[\partial_\mu \varphi_s\right](x_-) + \mathcal{O}\left(\lambda^2 \varphi_s\right) \tag{38}$$

of the soft field on the left-hand side of (33). This expansion of soft fields about the large coordinate $n_+ x$ of collinear fields must be applied whenever soft and collinear fields appear in a product. It generates an infinite tower of sub-leading in λ soft-collinear interactions, which precisely reproduces the expansion in the small soft momenta $n_{i+} k_s, k_{s\perp}$ of the momentum space amplitude. One may note the similarities to the standard multipole expansion for spatially localized systems, which also appears in non-relativistic effective theories.

对 (33) 式左侧软场的展开。每当软场和共线场出现在同一乘积中时，都需要对软场在共线场的大坐标 $n_+ x$ 处做这种展开。该展开会生成一个 λ 下无穷多的次领头软-共线相互作用层，这恰好重现了动量空间振幅在小软动量 $n_{i+} k_s, k_{s\perp}$ 下的展开。可以注意到它和空间局域系统标准多极展开的相似性，后者也出现在非相对论有效理论中。

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[5] Since the background field method may be more familiar in gauge theories, we invite the reader to compare the gauge transformations in SCET for QCD [6] with those for gravity discussed here.

[5] 由于背景场方法在规范理论中更为人熟知，我们请读者将本文讨论的引力规范变换，和 QCD 的 SCET 中规范变换 [6] 做对比。

This has an important implication for the gauge symmetry, since the gauge transformation (32) of $h_{\mu\nu}$ contains products of collinear and soft fields. Furthermore, the soft gauge transformation $U_s(x)$ of collinear fields is a product of soft and collinear fields, which still mixes different powers in λ , and therefore must be multipole-expanded. For example, the soft gauge transformation of the collinear matter field follows from (20) and reads

这对规范对称性有重要意义，因为 $h_{\mu\nu}$ 的规范变换 (32) 包含共线场与软场的乘积。此外，共线场的软规范变换 $U_s(x)$ 本身就是软场和共线场的乘积，它仍然会混合 λ 中的不同幂次，因此必须做多极展开。例如，共线物质场的软规范变换可由 (20) 得到，形式为

$$\varphi_c(x) \to U_s(x)\varphi_c(x) = U_s(x_-)\varphi_c(x) + x_\perp^\alpha \left[\partial_\alpha U_s\right](x_-)\varphi_c(x) + \dots . \tag{39}$$

The additional terms beyond the leading one imply that the transformation mixes different orders in λ in a way that is incompatible with the multipole expansion of the Lagrangian. To alleviate this, one needs to find a way to obtain collinear fields that have a homogeneous gauge transformation, respecting the multipole expansion. The resolution of this subtlety is technically quite involved, but intuitively very simple: recall that the multipole expansion is necessary, because soft fields cannot resolve the small-scale fluctuations of collinear modes. However, in the naive split $g_{\mu\nu}(x) = g_{s\mu\nu}(x) + \kappa h_{\mu\nu}(x)$, one implicitly assumes that this is possible, since the full $s_{\mu\nu}(x)$ appears inside $g_{s\mu\nu}(x)$. Therefore, the soft gauge symmetry, which is the symmetry of the background metric $g_{s\mu\nu}(x)$, is constructed with respect to the wrong background field. Instead, one needs to identify the appropriate background field $\hat{g}_{s\mu\nu}(x_-)$, whose constituent soft fields can only depend on x_-^μ . In other words, these soft background fields live only on the classical trajectory of the collinear particles. This proper background field comes with its residual transformation, and it is this residual transformation that is "homogeneous" in λ , respecting the multipole expansion.

领头项之外的附加项意味着该变换混合了 λ 的不同阶，这种方式与拉格朗日量的多极展开不兼容。为解决这个问题，我们需要找到一种方法得到满足多极展开、具有齐次规范变换的共线场。这个细节问题的解决在技术上相当复杂，但直观上非常简单: 回想多极展开之所以必要，是因为软场无法分辨共线模式的小尺度涨落。然而，在朴素拆分 $g_{\mu\nu}(x) = g_{s\mu\nu}(x) + \kappa h_{\mu\nu}(x)$ 中，我们隐含假设这是可以做到的，因为完整的 $s_{\mu\nu}(x)$ 出现在 $g_{s\mu\nu}(x)$ 内部。因此，作为背景度规 $g_{s\mu\nu}(x)$ 对称性的软规范对称性，是针对错误的背景场构造的。相反，我们需要确定合适的背景场 $\hat{g}_{s\mu\nu}(x_-)$ ，其组成软场只能依赖于 x_-^μ 。换言之，这些软背景场仅存在于共线粒子的经典轨迹上。这个正确的背景场带有自身的剩余变换，正是这个剩余变换对 λ 是“齐次”的，满足多极展开的要求。

A trace of this homogeneous background field can already be seen in the leading-power Lagrangian. By performing the multipole expansion and keeping only the leading terms in (5), one obtains the soft-collinear interaction

这种齐次背景场的痕迹已经可以在领头幂拉格朗日量中看到。通过做多极展开并仅保留 (5) 式中的领头项，我们得到软-共线相互作用

$$\mathcal{L}^{(0)} = -\frac{\kappa}{8} n_-^\mu n_-^\nu s_{\mu\nu}(x_-)\, n_+\partial\varphi_c n_+\partial\varphi_c. \tag{40}$$

When inserting this expression into a soft emission diagram, the combination $\varphi_c n_+\partial_+\varphi_c$ generates the eikonal propagator $\frac{in_+p}{2p\cdot k} = \frac{i}{n_-k}$. Therefore, the leading-power interaction from the homogeneous background field $s_{--}(x_-)$ yields a term proportional to $\varepsilon_{--}p_+$. Finding the full expression of the soft background field is more involved and is explained in section "Soft-Collinear Gravity." Once this proper background field is identified, it will be straightforward to construct the EFT systematically to all orders.

将该表达式代入软辐射图后，组合 $\varphi_c n_+\partial_+\varphi_c$ 产生了程函传播子 $\frac{in_+p}{2p\cdot k} = \frac{i}{n_-k}$ 。因此，齐次背景场 $s_{--}(x_-)$ 带来的领头幂次相互作用给出一个正比于 $\varepsilon_{--}p_+$ 的项。求解软背景场的完整表达式更为复杂，我们在“软共线引力”一节中进行说明。一旦确定了这个合适的背景场，就能很容易地系统构造出所有阶的有效场论。

Basic Features of the Effective Lagrangian

有效拉格朗日量的基本特征

The effective theory then takes a simple structure. The Lagrangian $\mathcal{L}_{\text{SCET}}$ splits into a soft-collinear and a purely soft Lagrangian

该有效理论有着简洁的结构。拉格朗日量 $\mathcal{L}_{\text{SCET}}$ 可分解为软共线拉格朗日量和纯软拉格朗日量

$$\mathcal{L}_{\text{SCET}} = \sum_i \mathcal{L}_{c_i}\left[h_{i\mu\nu}(x), s_{\mu\nu}(x_{i-})\right] + \mathcal{L}_s\left[s_{\mu\nu}(x)\right]. \tag{41}$$

In the collinear part, there is a sum over all collinear sectors defined by the directions n_{i-}^{μ} of the jets. This sum arises because one needs a hard scattering to generate particles of different collinear sectors. Since the Lagrangian does not describe hard scattering, there is no direct interaction vertex between different collinear sectors. Instead, these processes are allocated to the so-called N -jet operators, which generate energetic and soft particles from hard scattering.

在共线部分，需要对由喷注方向 n_{i-}^{μ} 定义的所有共线扇区求和。该求和的出现是因为，硬散射会产生不同共线扇区的粒子，而拉格朗日量不描述硬散射过程，因此不同共线扇区之间不存在直接相互作用顶点。这些过程反而被归到所谓的 N 喷注算符中，这类算符描述硬散射产生高能粒子与软粒子的过程。

The soft-collinear Lagrangian contains also purely collinear terms. This purely collinear Lagrangian, as well as the purely soft Lagrangian, is then completely equivalent to the original full theory (in weak-field expansion). This is due to the fact that if only one scale is present, e.g., by only considering purely collinear or purely soft modes without any external sources, then there is no Lorentz-invariant notion of soft or collinear. One could simply perform a Lorentz boost and collinear modes would become soft, and vice versa. It is the presence of a source that provides meaning to the notion of soft and collinear in the first place.

软共线拉格朗日量也包含纯共线项。在弱场展开下，该纯共线拉格朗日量与纯软拉格朗日量都完全等价于原始的完整理论。这是因为，如果仅存在一个能标，例如仅考虑纯共线模式或纯软模式、不存在任何外源，那么洛伦兹不变性下软和共线的定义不成立: 只需做一个洛伦兹 boost，共线模式就能变为软模式，反之亦然。源的存在才首先让软和共线的概念变得有意义。

Therefore, all the non-trivial physics is contained inside the soft-collinear interaction vertices as well as the ”sources,” which are described by N -jet operators. These vertices stem from the terms in the collinear Lagrangian that are covariant with respect to the non-trivial soft background.

因此，所有非平庸物理都包含在软共线相互作用顶点以及由 N 喷注算符描述的“源”中。这些顶点来源于共线拉格朗日量中，关于非平庸软背景协变的项。

From this perspective, the following structure of the theory arises, shown in Fig. 3. The collinear sector i is constructed to be covariant with respect to a soft background metric denoted by $\hat{g}_{s_i\mu\nu}(x_{i-})$, which is constructed from the dynamical soft field $\eta_{\mu\nu} + \kappa s_{\mu\nu}(x)$, restricted to the classical light-like trajectories x_{i-}^{μ} of the energetic particles. Since due to the multipole expansion, collinear fields interact with soft fields at x_{i-}^{μ} only, these are effectively $i = 1, \ldots, N$ separate soft gauge symmetries $U_s(x_{i-})$. Note that the interactions with the soft field are blind to the non-locality of the collinear sector in the n_{i+}^{μ} direction since $\left(x^{\mu} + t n_{i+}^{\mu}\right)_- = x_-^{\mu}$. On the other hand, the soft fields have self-interactions, which pervade all of spacetime, and are controlled by the soft Lagrangian $\mathcal{L}_s$.

从该视角出发, 我们得到理论的如下结构, 如图 3 所示。共线扇区 i 被构造为对软背景度规 $\hat{g}_{s_i\mu\nu}(x_{i-})$ 协变，该度规由动力学软场 $\eta_{\mu\nu}+\kappa s_{\mu\nu}(x)$ 构造，限制在高能粒子的经典类光轨迹 x_{i-}^{μ} 上。由于多极展开，共线场仅在 x_{i-}^{μ} 处与软场相互作用，因此这实际上是 $i=1,\ldots,N$ 个独立的软规范对称性 $U_s(x_{i-})$。注意，由于 $\left(x^{\mu}+tn_{i+}^{\mu}\right)_{-}=x_{-}^{\mu}$，与软场的相互作用对共线扇区在 n_{i+}^{μ} 方向的非局域性不敏感。另一方面，软场存在自相互作用，这类相互作用遍布整个时空，由软拉格朗日量 $\mathcal{L}_s$ 描述。

Gravity vs QCD: A Comparison

引力 vs 量子色动力学: 对比

At this point, it is instructive to compare the gravitational situation to the gauge theory one, where the SCET construction is well understood. The basic aspects of the construction, as discussed in section ”Basic Concepts of SCET,” are the same for both theories, replacing the background field $\hat{g}_{s\mu\nu}(x_-)\to n_-A_s(x_-)$.

至此，将引力场景与 SCET 构造已被充分理解的规范理论场景进行对比会颇有启发性。正如“SCET 基本概念”一节所讨论，该构造的基本方面对两种理论是相同的，只需替换背景场 $\hat{g}_{s\mu\nu}(x_-)\to n_-A_s(x_-)$。

The first main difference lies in the nature of the ”full theory” itself. In QCD, the starting point for the EFT construction is Yang-Mills theory, a renormalizable field theory. Contrast this with gravity: here, the natural starting point, Einstein-Hilbert theory, is not renormalizable. Instead, one takes the effective action (7), up to a desired order in the loop expansion (or curvature, respectively). In addition, one then performs the weak-field expansion, since the relevant degree of freedom is the fluctuation $h_{\mu\nu}$, the graviton field. Therefore, the ”full theory” underlying SCET gravity is the weak-field expansion of an effective extension of the Einstein-Hilbert action and thus already defined as an infinite series in κ that must be truncated at some order. This truncated theory is then expanded in λ . The λ -expansion shares many features with the κ -expansion but is not identical to it.

第一个主要差异在于“完整理论”本身的性质。在量子色动力学中，EFT 构造的出发点是杨-米尔斯理论，这是一种可重整化量子场论。与之相比，引力的自然出发点爱因斯坦-希尔伯特理论是不可重整化的。我们需要取有效作用量 (7)，保留圈展开 (或曲率) 中到目标阶数的项。此外，我们随后要做弱场展开，因为相关自由度是涨落 $h_{\mu\nu}$，即引力子场。因此，SCET 引力所基于的“完整理论”是爱因斯坦-希尔伯特作用量有效推广的弱场展开，本身就定义为 κ 中的无穷级数，必须在某一阶截断。该截断理论随后会按 λ 展开。λ 展开与 κ 展开有诸多共性，但并不完全相同。

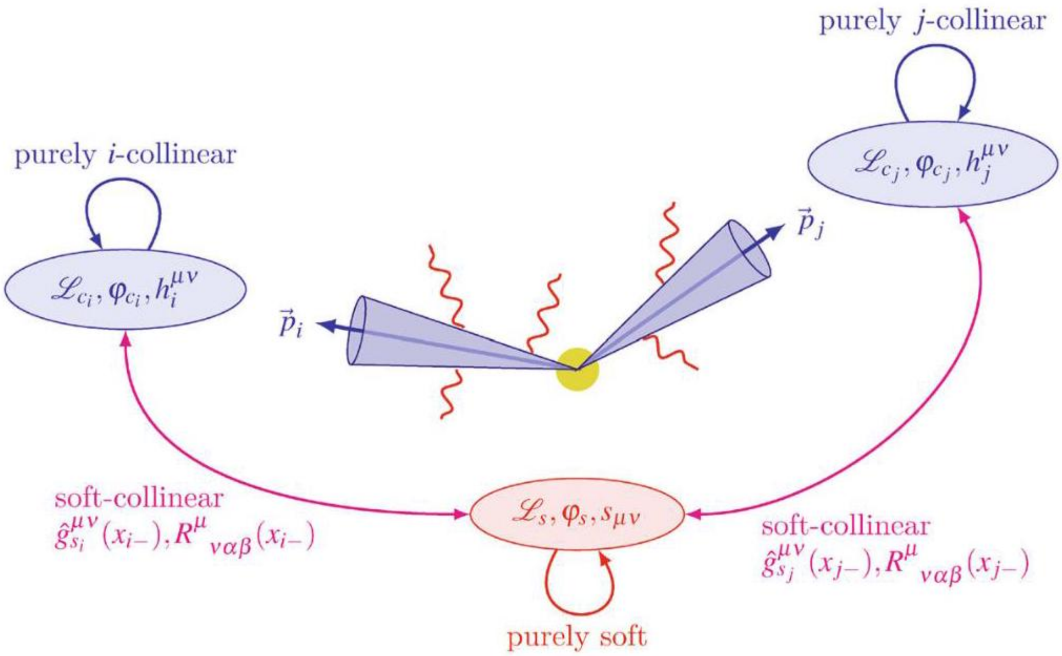

Fig. 3 An intuitive picture of the form of SCET. The soft modes (red) are described by the purely soft Lagrangian $\mathcal{L}_s$, which takes the same form as the full theory. Each collinear sector (blue) is described by its own Lagrangian $\mathcal{L}_{c_i}$, which contains purely collinear and soft-collinear interactions (pink) These soft-collinear terms are covariant with respect to a homogeneous background field $\hat{g}_{s_i}^{\mu\upsilon}(x_{i-})$, living only on the classical trajectory x_{i-}^{μ} of the collinear particles, and describe soft-collinear physics to all orders

图 3 SCET 形式的直观示意图。软模式 (红色) 由纯软拉格朗日量 $\mathcal{L}_s$ 描述，其形式与完整理论一致。每个共线扇区 (蓝色) 由各自的拉格朗日量 $\mathcal{L}_{c_i}$ 描述，其中包含纯共线相互作用与软-共线相互作用 (粉色)。这些软-共线项关于均匀背景场 $\hat{g}_{s_i}^{\mu\upsilon}(x_{i-})$ 协变，该背景场仅存在于共线粒子的经典轨迹 x_{i-}^{μ} 上，对软-共线物理的描述包含了所有阶贡献

Another major difference is the nature of the underlying gauge symmetry. In QCD, the gauge symmetry is purely internal, and it has a color charge with a corresponding generator $t^a \sim \lambda^0$ that is unrelated to the kinematics that define the SCET expansion. In gravity, however, the charge is related to momenta P^{μ} , and therefore, the gauge symmetry is connected to kinematics. In particular, the collinear momentum components have different λ -scaling (23). This implies that the gauge transformation parameters in gravity also have non-trivial power counting and are not homogeneous in λ . For example, a collinear matter field transforms under an infinitesimal collinear transformation as

另一个主要差异是底层规范对称性的性质。在量子色动力学中，规范对称性完全是内禀的，它带有带对应生成元 $t^a \sim \lambda^0$ 的色荷，该生成元与定义 SCET 展开的运动学无关。但在引力中，荷与动量 P^{μ} 相关，因此规范对称性与运动学直接相关。尤其是，共线动量分量具有不同的 λ 标度 (23)。这意味着引力中的规范变换参数也具有非平庸幂计数，在 λ 下不是齐次的。例如，共线物质场在无穷小微共线变换下的变换为

$$\varphi_c \to \varphi_c - \kappa \varepsilon_c^{\alpha} \partial_{\alpha} \varphi_c \tag{42}$$

where the second term is suppressed by $\mathcal{O}(\lambda)$ relative to the first due to the scaling of ∂_μ and the gauge parameter (see (45) below). Therefore, gauge transformations mix different powers of λ , and no homogeneous scaling can be achieved when manifest gauge invariance is imposed. An object can be either homogeneous in λ or gauge-invariant, but never both at the same time.

> 由于 ∂_μ 和规范参数的标度 (见下文 (45) 式)，第二项相对于第一项被 $\mathcal{O}(\lambda)$ 压低。因此，规范变换会混合 λ 的不同次幂，要求显式规范不变性时无法得到齐次标度。一个对象要么在 λ 下齐次，要么满足规范不变性，二者不可能同时成立。

These formal differences aside, one can now take a look at the physical content of both theories. In QCD, the collinear gluon field A_c contains two physical and two unphysical components. The large component $n_+A_c \sim \lambda^0$ and the small $n_-A_c \sim \lambda^2$ are unphysical, while the transverse components $A_{c\perp} \sim \lambda$ are physical. The large component n_+A_c is problematic: if this field was an allowed building block, one could add arbitrarily many such fields to any operator while keeping its λ -counting fixed. Thus, there would not exist a finite operator basis, and the power counting would be meaningless, since one would have to perform an infinite number of matching computations already at leading power. This is alleviated by noting that in light cone gauge, $n_+A_c = 0$; thus, it is a gauge-artefact. Therefore, one introduces a Wilson line W_c (definition in (47) below), which controls these n_+A_c to all orders. These Wilson lines are homogeneous in λ , but an infinite series in the gauge coupling g_s . In gravity, a similar situation arises, but worse with respect to the power counting: the collinear graviton $h_{\mu\nu}$ contains modes $h_{++} \sim \lambda^{-1}$ and $h_{+\perp} \sim \lambda^0$, where h_{++} is even power counting enhanced. Clearly, these must be controlled. Therefore, one employs a similar concept as in QCD, by introducing the analogue of the collinear Wilson line. In this way, the unphysical $h_{+\mu}$ components can be controlled to all orders. Just as in QCD, this ”Wilson line” is an infinite series in the coupling κ . Due to the aforementioned inhomogeneity of the gauge symmetry, however, this implies that these gravitational ”Wilson lines” are also an infinite series in λ and no longer homogeneous. These ”Wilson lines” can be used to implement a covariant version of light cone gauge, in the sense that one defines gauge-invariant composite objects that satisfy the light cone gauge properties.

> 撇开这些形式上的差异不谈，现在我们可以考察两种理论的物理内容。在量子色动力学中，共线胶子场 A_c 包含两个物理分量和两个非物理分量。大分量 $n_+A_c \sim \lambda^0$ 和小分量 $n_-A_c \sim \lambda^2$ 是非物理的，而横分量 $A_{c\perp} \sim \lambda$ 是物理的。大分量 n_+A_c 存在问题: 如果该场是可允许的构造单元，那么我们可以向任意算符中添加任意多个这类场，同时保持其 λ 计数不变。这样一来就不存在有限的算符基，幂次计数也将失去意义，因为哪怕是领头阶也需要完成无穷多次匹配计算。这个问题可以通过下述结论解决: 在光锥规范下，$n_+A_c = 0$，因此这个分量是规范规范带来的伪影。为此我们引入威尔逊线 W_c (定义见下文 (47) 式)，它可以对所有阶的 n_+A_c 进行约束。这些威尔逊线对 λ 是齐次的，但对规范耦合 g_s 是无穷级数展开。在引力中也存在类似情况，但幂次计数方面的问题更严重: 共线引力子 $h_{\mu\nu}$ 包含模式 $h_{++} \sim \lambda^{-1}$ 和 $h_{+\perp} \sim \lambda^0$，其中 h_{++} 甚至是幂次计数增强的。显然这些模式必须得到约束。因此我们采用了和量子色动力学中类似的思路，引入共线威尔逊线的对应构造。通过这种方法，我们可以对所有阶的非物理 $h_{+\mu}$ 分量进行约束。和量子色动力学一样，这种“威尔逊线”是耦合常数 κ 的无穷级数展开。但由于前面提到的规范对称性的非齐次性，这意味着引力中的这类“威尔逊线”同时也是 λ 的无穷级数展开，不再是齐次的。这些“威尔逊线”可以用来实现光锥规范的协变版本，具体来说，我们用它定义满足光锥规范性质的规范不变复合对象。

The suppressed unphysical degrees of freedom, n_-A_c in QCD and $h_{-\perp}, h_{-\perp}$ in gravity, can be eliminated using the equations of motion. This leaves only the two transverse polarizations $A_{c\perp} \sim \lambda$ and $h_{\perp\perp} \sim \lambda$ as physical degrees of freedom in the effective theory. Since they count with a positive power of λ , a sensible operator basis exists.

被压制的非物理自由度——量子色动力学中的 $n_{-}A_c$ 和引力中的 $h_{-\perp}, h_{-\perp}$ ——可以通过运动方程消去。有效理论中仅剩下两个横极化 $A_{c\perp} \sim \lambda$ 和 $h_{\perp\perp} \sim \lambda$ 作为物理自由度。由于它们都是 λ 的正幂次，因此存在合理的算符基。

The soft sector is slightly different from the collinear one. Here, one first needs to identify the appropriate homogeneous background field, and then one organizes the sub-leading terms in gauge-covariant objects. The background field comes with a covariant derivative, which can be eliminated in the sources and the sub-leading Lagrangian terms using the equations of motion. For the operator basis, only the sub-leading gauge-covariant objects are relevant. In gauge theory, this gauge-covariant object is the field strength tensor $F_{s\mu\nu} \sim \lambda^4$, which is the first derivative of the gluon field. In gravity, the first sensible gauge-covariant object is the Riemann tensor, $R^{\mu}{}_{\nu\alpha\beta} \sim \lambda^6$, which is the second derivative of the metric fluctuation. These gauge-covariant objects can appear in the sources and mediate process-dependent soft emissions. Therefore, already at this stage, one can anticipate that in a soft emission process, there are universal terms and the non-universality in gravity is more strongly suppressed compared to gauge theory. The previously discussed features are summarized in Table 1. While there are essential differences between both theories, the construction of effective theories proceeds similarly. This will guide the following sections.

软扇区和共线扇区略有不同。在这里我们首先需要找到合适的齐次背景场，再将次领头项组织到规范协变对象中。背景场带有协变导数，可以利用运动方程将其从源和次领头拉格朗日项中消去。对于算符基而言，只有次领头的规范协变对象是相关的。规范理论中，这个规范协变对象是场强张量 $F_{s\mu\nu} \sim \lambda^4$ ，它是胶子场的一阶导数。引力中，第一个合理的规范协变对象是黎曼张量 $R^{\mu}{}_{\nu\alpha\beta} \sim \lambda^6$ ，它是度规涨落的二阶导数。这些规范协变对象可以出现在源中，传递依赖过程的软辐射。因此在这个阶段我们就可以预期，软辐射过程中存在普适项，且相比规范理论，引力中的非普适性压制得更强。前文讨论的特征总结在表 1 中。尽管两种理论存在本质差异，但有效理论的构造流程是相似的，这将指导后续章节的讨论。

Table 1 A comparison of the main features of gauge theory (QCD) and gravity from the SCET perspective

表 1 从软共线有效理论视角比较规范理论 (量子色动力学) 与引力的主要特征

	QCD	Gravity
Gauge symmetry	SU(3)	Diff (M)
Gauge charge	$t^a \sim \lambda^0$	$P^\mu \sim (\lambda^0, \lambda, \lambda^2)$
Dimensionful coupling	no	yes
Fundamental degree of freedom	$A_\mu \sim p_\mu$	$h_{\mu\nu} \sim \frac{p_\mu p_\nu}{\lambda}$
Unsuppressed components	$n_+ A_c \sim 1$	$h_{++} \sim \lambda^{-1}, h_{+\perp} \sim 1$
Physical degrees of freedom	$A_{c\perp} \sim \lambda$	$h_{\perp\perp} \sim \lambda$
Redundant degrees of freedom	$n_- A_c \sim \lambda^2$	$h_{\perp -} \sim \lambda^2, h_{--} \sim \lambda^3$
Field strength/curvature	$F_{\mu\nu} \sim \partial A$	$R^{\mu}{}_{\nu\alpha\beta} \sim \partial^2 h$

Collinear Gravity

共线引力

It is convenient to first consider only collinear modes in order to familiarize oneself with the construction. The purely collinear EFT for gravity is obtained after splitting a generic field $\phi_J = \phi_{J,h} + \sum_i \phi_{J,c_i}$. After integrating out the hard modes, there are no leftover interactions of the collinear fields ϕ_{J,c_i} of different sectors, as the sum of the collinear momenta belonging to different directions has hard scaling and corresponds to off-shell degrees of freedom which have already been eliminated. The Lagrangian is simply the sum

为了熟悉构造过程，先仅考虑共线模式会更为方便。拆分一般场 $\phi_J = \phi_{J,h} + \sum_i \phi_{J,c_i}$ 后即可得到引力的纯共线有效场论。积出硬模式后，不同扇区的共线场 ϕ_{J,c_i} 不存在剩余相互作用，因为属于不同方向的共线动量之和满足硬标度，对应已经被消除的离壳自由度。拉格朗日量简单地可写为求和形式

$$\mathcal{L}_{\text{SCET}} = \sum_i \mathcal{L}_{c_i} \tag{43}$$

For this reason, it is possible to focus only on a single collinear direction. To simplify the notation, in the following, the subscripts c and i are omitted. In addition to the Lagrangian, the theory contains so-called currents (aka "sources" or " N -jet operators"), which contain products of fields in more than one collinear direction. Their matching coefficients absorb the hard modes that have been integrated out. These are discussed in more detail in section "Sources and Hard Matching." Here, we focus on the effective Lagrangian.

因此，我们可以仅聚焦于单个共线方向。为简化记号，下文将省略下标 c 和 i。除拉格朗日量外，该理论还包含所谓的流 (也叫“源”或“N 喷注算符”)，这类流包含多个共线方向的场乘积，它们的匹配系数吸收了已被积出的硬模式。我们会在“源与硬匹配”一节中详细讨论，这里我们聚焦于有效拉格朗日量。

At this stage, the gauge symmetry consists of N -copies of collinear gauge symmetry, such that each collinear sector transforms under its own gauge symmetry and is invariant with respect to all the remaining collinear symmetries. Collinear gauge transformations must not distort the scaling of the collinear gauge fields. In contrast with SCET QCD, in SCET gravity, the gauge transformation parameter ε^μ acquires λ -scaling. Enforcing homogeneity of the infinitesimal gauge transformation

在此阶段，规范对称性由 N 份共线规范对称性构成，每个共线扇区在自身的规范对称性下变换，且对所有其余共线对称性保持不变。共线规范变换不能改变共线规范场的标度。与 SCET 量子色动力学不同，在 SCET 引力中，规范变换参数 ε^μ 具有 λ 标度。要求无穷小规范变换满足齐次性

$$h_{\mu\upsilon} \to h_{\mu\upsilon} - \partial_\mu \varepsilon_\upsilon - \partial_\upsilon \varepsilon_\mu \sim \mathcal{O}\left(h_{\mu\upsilon}\right), \tag{44}$$

ε^μ scales as

ε^μ 的标度为

$$n_+\varepsilon \sim \frac{1}{\lambda},\ n_-\varepsilon \sim \lambda,\ \varepsilon^{\mu_\perp} \sim 1. \tag{45}$$

Note that ε^μ scales as a small translation λx^μ . Beyond the linear order, the transformation of $h_{\mu\nu}$ takes the form

注意 ε^μ 的标度与小平移 λx^μ 一致。在线性阶以上，$h_{\mu\nu}$ 的变换形式为

$$h'_{\mu\upsilon} = h_{\mu\upsilon} - \partial_\mu\varepsilon_\upsilon - \partial_\upsilon\varepsilon_\mu - \kappa\left[\partial_\mu\varepsilon^\alpha h_{\alpha\upsilon} - \partial_\upsilon\varepsilon^\alpha h_{\alpha\mu} - \varepsilon^\alpha\partial_\alpha h_{\mu\upsilon}\right. \tag{46}$$

$$\left.+\partial_\mu\varepsilon^\alpha\partial_\alpha\varepsilon_\upsilon + \partial_\upsilon\varepsilon^\alpha\partial_\alpha\varepsilon_\mu + \partial_\mu\varepsilon_\alpha\partial_\upsilon\varepsilon^\alpha + \varepsilon^\alpha\partial_\alpha\left(\partial_\mu\varepsilon_\upsilon + \partial_\upsilon\varepsilon_\mu\right)\right] + \mathcal{O}\left(\lambda^3\right).$$

Collinear gauge invariance not only is required from a formal point of view but also ensures that the EFT power counting is meaningful, i.e., one cannot generate an infinite number of operators with the help of the unsuppressed collinear fields. For this reason, it is beneficial to introduce the concept of gauge-invariant building blocks. An arbitrary current contains fields in multiple collinear directions. Since each sector has its own gauge symmetry, the invariance of the complete operator under each collinear symmetry implies that each sector is separately gauge-invariant. This is automatically achieved if the operator is built from the gauge-invariant building blocks defined as the collinear field dressed with ”Wilson lines,” chosen such that the building block is a gauge singlet that is always suppressed by at least one power of λ . In gauge theory, this idea leads to the concept of the collinear Wilson line W_c

共线规范不变性不仅是形式上的要求，还保证了有效场论幂计数是有意义的，即不会利用未压低的共线场生成无穷多个算符。因此，引入规范不变构造块的概念是有益的。任意流包含多个共线方向的场，由于每个扇区有自身的规范对称性，完整算符对每个共线对称性的不变性意味着每个扇区单独满足规范不变性。若算符由规范不变构造块构建，就能自动满足这一点: 这些构造块定义为缀饰了“威尔逊线”的共线场，经选取后构造块是规范单态，且至少总会被一个 λ 幂次压低。在规范理论中，这一思想引出了共线威尔逊线 W_c 的概念

$$W_c(x) = P\exp\left(ig_s\int_{-\infty}^{0} dsn_+A_c(x+sn_+)\right), \tag{47}$$

which fulfils the following identity:

它满足如下恒等式:

$$W_c^\dagger n_+D_cW_c = n_+\partial, \tag{48}$$

such that the collinear gauge-invariant gluon building block is $\mathcal{A}_\mu = \frac{1}{g_s}W_c^\dagger\ \left[iD_{c\mu}W_c\right]$. This leads to the notion of ”covariant light cone gauge,” since the composite operator $\mathcal{A}_\mu$ built from $A_{c\mu}$ satisfies $n_+\mathcal{A} = 0$ in any gauge.

因此共线规范不变胶子构造块为 $\mathcal{A}_\mu = \frac{1}{g_s}W_c^\dagger\ \left[iD_{c\mu}W_c\right]$。这引出了“协变光锥规范”的概念，因为由 $A_{c\mu}$ 构造的复合算符 $\mathcal{A}_\mu$ 在任意规范下都满足 $n_+\mathcal{A} = 0$。

In analogy to QCD, where the collinear gauge-invariant gluon field satisfies $\mathcal{A}_+ = 0$, in SCET gravity, one constructs the manifestly gauge-invariant graviton building block $\mathfrak{h}_{\mu\upsilon}$ [11, 25, 35] , which depends on the collinear field $h_{\mu\upsilon}$, and satisfies $\mathfrak{h}_{\mu+} = 0$, that is, it coincides with the elementary graviton field in light cone gauge. Just like in QCD, where the collinear Wilson line connects the collinear gluon field to the gauge-invariant object, in gravity, the collinear graviton field is related to the gauge-invariant graviton building block via

类比量子色动力学中共线规范不变胶子场满足 $\mathcal{A}_+ = 0$ 的情况，在 SCET 引力中可以构造明显规范不变的引力子构造块 $\mathfrak{h}_{\mu\upsilon}$ [11, 25, 35]，它依赖共线场 $h_{\mu\upsilon}$ 且满足 $\mathfrak{h}_{\mu+} = 0$，即与光锥规范下的基本引力子场一致。就像量子色动力学中，共线威尔逊线将共线胶子场与规范不变对象联系起来，在引力中，共线引力子场与规范不变引力子构造块也通过下式关联:

$$\eta_{\mu\nu} + \kappa\mathfrak{h}_{\mu\nu}(x) = W_\mu^\alpha W_\nu^\beta \left[W_c^{-1} g_{\alpha\beta}(x)\right], \tag{49}$$

where $g_{\alpha\beta}(x) = \eta_{\alpha\beta} + \kappa h_{\alpha\beta}(x)$ and the gravitational collinear "Wilson line" is

其中 $g_{\alpha\beta}(x) = \eta_{\alpha\beta} + \kappa h_{\alpha\beta}(x)$，引力共线“威尔逊线”为

$$W_c^{-1} = T_{\theta_c}[h] = 1 + \kappa\theta_c^\alpha\partial_\alpha + \frac{\kappa^2}{2}\theta_c^\alpha\theta_c^\beta\partial_\alpha\partial_\beta + \mathcal{O}\left(\theta_c^3\right), \tag{50}$$

with parameter $\theta_c[h] \equiv \theta_c\left[h_{\mu\upsilon}(x)\right]$ chosen such that, given $h_{\mu\upsilon}$, the invariant field $\mathfrak{h}_{\mu\nu}$ satisfies $\mathfrak{h}_{\mu+} = 0$, and W_μ^α is the Jacobian for the transformation (50), defined as in (14). The above translation corresponds to the transformation to the coordinate system, in which the metric fluctuation satisfies light cone gauge. Note, however, that we do not actually fix the gauge. Rather, W_c is employed to define the composite operator $\mathfrak{h}_{\mu\upsilon}(x)$ built from the elementary field $h_{\mu\upsilon}$ for which no special coordinate system or gauge fixing is assumed. The "Wilson line" transforms as [25]

选取参数 $\theta_c[h] \equiv \theta_c\left[h_{\mu\upsilon}(x)\right]$ 使得，给定 $h_{\mu\upsilon}$，不变场 $\mathfrak{h}_{\mu\nu}$ 满足 $\mathfrak{h}_{\mu+} = 0$，且 W_μ^α 是变换 (50) 的雅可比行列式，定义同 (14)。上述平移对应到满足度规涨落光锥规范的坐标系的变换。但需要注意，我们实际上并没有固定规范。相反，我们采用 W_c 来定义由基本场 $h_{\mu\upsilon}$ 构造的复合算符 $\mathfrak{h}_{\mu\upsilon}(x)$，对该基本场不假设特殊的坐标系或规范固定。“威尔逊线”的变换为 [25]

$$W_c^{-1} \to W_c^{-1}U^{-1}(x) \tag{51}$$

and this transformation precisely cancels the collinear gauge transformation of $h_{\mu\upsilon}$ in (49). It is also used to define the collinear gauge-invariant matter field χ_c ,

该变换恰好抵消了 (49) 中 $h_{\mu\upsilon}$ 的共线规范变换。它还被用于定义共线规范不变物质场 χ_c，

$$\chi_c = \left[W_c^{-1}\varphi_c\right] = \varphi_c + \mathcal{O}(\lambda\varphi_c). \tag{52}$$

Note that $W_c^{-1} = 1 + \mathcal{O}(\lambda)$ in stark contrast to (47), which is closely related to the absence of purely collinear gravitational interactions at leading power in λ as will be seen below.

请注意，$W_c^{-1} = 1 + \mathcal{O}(\lambda)$ 与 (47) 截然不同，这与下文将会看到的，在 λ 领头幂次下不存在纯共线引力相互作用密切相关。

The explicit construction of the collinear ”Wilson line” requires determining the parameter $\theta_c^\mu[h]$. The closed-form version does not exist in gravity, but instead it can be defined perturbatively in the weak-field expansion

共线“威尔逊线”的显式构造需要确定参数 $\theta_c^\mu[h]$。引力中不存在闭合形式，不过它可以在弱场展开下按微扰定义

$$\theta_c^\mu = \theta_c^{\mu(0)} + \theta_c^{\mu(1)} + \ldots \tag{53}$$

Then, one needs to perform the expansion of (49),

接下来，需要对 (49) 做展开，

$$\mathfrak{h}_{\mu\upsilon} = h_{\mu\upsilon} + \partial_\mu\theta_{c\upsilon} + \partial_\upsilon\theta_{c\mu} + \mathcal{O}\left(\theta_c^2, \theta_c h\right), \tag{54}$$

and insert (53). Requiring $\mathfrak{h}_{\mu+} = 0$ determines

再代入 (53)。由要求 $\mathfrak{h}_{\mu+} = 0$ 可确定

$$\theta_{c\mu}^{(0)} = -\frac{1}{n_+\partial}\left(h_{\mu+} - \frac{1}{2}\frac{\partial_\mu}{n_+\partial}h_{++}\right). \tag{55}$$

The sub-leading (non-linear) terms can be obtained iteratively [11]. The inverse derivative is defined as

次领头 (非线性) 项可以通过迭代得到 [11]。逆导数定义为

$$\frac{1}{in_+\partial + i\varepsilon}f(x^\mu) = -i\int_{-\infty}^0 ds f\left(x^\mu + sn_+^\mu\right). \tag{56}$$

Inserting back the leading term in the expansion of θ_c (55) into $\mathfrak{h}_{\mu\upsilon}$ (54) leads to the explicit formula for the gauge-invariant graviton building block expressed in terms of the collinear graviton

将 θ_c 展开式 (55) 中的领头项代回 $\mathfrak{h}_{\mu\nu}$ (54)，可得用共线引力子表示的规范不变引力子构造块的显式公式

$$\mathfrak{h}_{\mu\upsilon} = h_{\mu\upsilon} - \frac{\partial_\mu}{n_+\partial}\left(h_{\upsilon+} - \frac{1}{2}\frac{\partial_\upsilon}{n_+\partial}h_{++}\right) - \frac{\partial_\upsilon}{n_+\partial}\left(h_{\mu+} - \frac{1}{2}\frac{\partial_\mu}{n_+\partial}h_{++}\right) + \mathcal{O}\left(\lambda h_{\mu\upsilon}\right),$$

(57)

which resembles the analogous expression for the gauge-invariant collinear gluon field, [6]

其形式与规范不变共线胶子场的对应表达式类似，[6]

$$\mathcal{A}_{c\mu_\perp} = \frac{1}{g_s} W_c^\dagger \left[iD_{c\mu_\perp} W_c \right] = A_{c\mu_\perp} - \frac{\partial_{\mu_\perp}}{n_+ \partial} A_{c+} + \mathcal{O}\left(g_s A_{c\mu_\perp}\right). \tag{58}$$

Having understood the basic building blocks of the theory, it is straightforward to derive the collinear Lagrangian. The first step involves performing the weak-field expansion of (5). Then one re-expresses the collinear fields in terms of the gauge-invariant building blocks by inserting the gravitational collinear "Wilson line" (50). Since the "Wilson line" corresponds to a collinear gauge transformation, one can simply put $h_{\mu\upsilon} \to \mathfrak{h}_{\mu\upsilon}$ and $\varphi_c \to \chi_c$. Then, one can set $\mathfrak{h}_{\mu+} = 0$. Finally, grouping the terms by their order in the λ -expansion yields

理解了该理论的基本构造块后，推导共线拉格朗日量就十分直接了。第一步是对 (5) 做弱场展开，再代入引力共线“威尔逊线”(50)，将共线场重新表示为规范不变构造块的形式。由于“威尔逊线”对应一个共线规范变换，我们可以直接令 $h_{\mu\upsilon} \to \mathfrak{h}_{\mu\upsilon}$ 和 $\varphi_c \to \chi_c$ 。接着可以设 $\mathfrak{h}_{\mu+} = 0$ 。最后，按各项在 λ 展开中的幂次整理得到

$$\mathcal{L}^{(0)} = \frac{1}{2} \partial_\mu \chi_c \partial^\mu \chi_c \tag{59}$$

$$\mathcal{L}^{(1)} = -\frac{\kappa}{2} \left(\mathfrak{h}_{\mu\nu} \partial^\mu \chi_c \partial^\nu \chi_c - \frac{1}{2} \mathfrak{h} \partial_\alpha \chi_c \partial^\alpha \chi_c \right), \tag{60}$$

$$\mathcal{L}^{(2)} = \frac{\kappa^2}{2} \left(\mathfrak{h}^{\mu\alpha} \mathfrak{h}_\alpha^\upsilon - \frac{1}{2} \mathfrak{h} \mathfrak{h}^{\mu\upsilon} + \frac{1}{8} \left(\mathfrak{h}^2 - 2 \mathfrak{h}^{\alpha\beta} \mathfrak{h}_{\alpha\beta} \right) \eta^{\mu\upsilon} \right) \partial_\mu \chi_c \partial_\upsilon \chi_c. \tag{61}$$

The leading-power Lagrangian $\mathcal{L}^{(0)}$ describes a free theory of gauge-invariant building blocks. It follows from this simple observation that there cannot be collinear singularities in gravity. Interactions with collinear gravitons start at sub-leading power, and all terms are manifestly gauge-invariant.

领头幂次拉格朗日量 $\mathcal{L}^{(0)}$ 描述了规范不变构造块的自由理论。从这个简单观察可以推出，引力中不可能存在共线奇点。与共线引力子的相互作用从次领头幂次开始，且所有项都是明显规范不变的。

The Lagrangian above is expressed entirely in terms of the gauge-invariant building blocks χ_c and $\mathfrak{h}_{\mu\nu}$. This form makes it clear that there is a well-defined expansion in λ , since the metric components unsuppressed in λ do not appear due to $\mathfrak{h}_{\mu+} = 0$. Since the Lagrangian is collinear gauge-invariant, it takes the same form when expressed in terms of the original fields φ_c and $h_{\mu\nu}$. However, since h_{++} and $h_{+\perp}$ do not vanish, this form hides the gauge cancellations that occur between the unsuppressed metric components and lacks manifest power counting.

上述拉格朗日量完全由规范不变构造块 χ_c 和 $\mathfrak{h}_{\mu\nu}$ 表示。这种形式清楚表明，对 λ 存在定义良好的展开，因为由于 $\mathfrak{h}_{\mu+} = 0$ ，未在 λ 中压低的度规分量不会出现。由于该拉格朗日量是共线规范不变的，因此用原场 φ_c 和 $h_{\mu\nu}$ 表示时形式不变。但由于 h_{++} 和 $h_{+\perp}$ 并不为零，这种形式掩盖了未压低度规分量之间发生的规范抵消，也不具有明显幂计数。

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[6] Despite the fact that $W_c^{-1} = 1 + \mathcal{O}(\lambda)$, $\mathfrak{h}_{\mu\upsilon}$ differs from $h_{\mu\upsilon}$ at $\mathcal{O}\left(\lambda^0 h_{\mu\upsilon}\right)$ due to the Jacobian terms in (49) multiplying the $\eta_{\alpha\beta}$ term in $g_{\alpha\beta}(x)$.

[6] 尽管由于 (49) 中乘在 $g_{\alpha\beta}(x)$ 里 $\eta_{\alpha\beta}$ 项上的雅可比项，$W_c^{-1} = 1 + \mathcal{O}(\lambda)$, $\mathfrak{h}_{\mu\nu}$ 在 $\mathcal{O}(\lambda^0 h_{\mu\nu})$ 阶与 $h_{\mu\nu}$ 不同。

The sub-leading components $\mathfrak{h}_{\mu-}$ can be eliminated from the Lagrangian by using equations of motion. To their respective leading order in λ , one finds [11]

次领头分量 $\mathfrak{h}_{\mu-}$ 可以通过运动方程从拉格朗日量中消去。在各自的 λ 领头阶下，可得文献 [11] 中的结果:

$$\mathfrak{h}_{\mu_\perp -} = -2\frac{\partial^{\alpha_\perp}}{n_+\partial}\mathfrak{h}_{\mu_\perp\alpha_\perp} + \mathcal{O}(\lambda^3), \tag{62}$$

$$\mathfrak{h}_{--} = 4\frac{\partial^{\alpha_\perp}\partial^{\beta_\perp}}{(n_+\partial)^2}\mathfrak{h}_{\alpha_\perp\beta_\perp} + \mathcal{O}(\lambda^4), \tag{63}$$

where the first equation arises from the field equation for $\mathfrak{h}_{\mu_\perp -}$ and the second from tracing the one for $\mathfrak{h}_{\mu_\perp\nu_\perp}$. The Lagrangian is linear in $\mathfrak{h}_{--}$, and the field equation for $\mathfrak{h}_{--}$ results in the trace constraint

其中第一个方程由 $\mathfrak{h}_{\mu_\perp -}$ 的场方程得到，第二个方程由对 $\mathfrak{h}_{\mu_\perp\nu_\perp}$ 的场方程求迹得到。拉格朗日量对 $\mathfrak{h}_{--}$ 是线性的，因此 $\mathfrak{h}_{--}$ 的场方程给出迹约束

$$\mathfrak{h} = \frac{\kappa}{2}\left(\mathfrak{h}_{\alpha_\perp\beta_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp} - \frac{1}{(n_+\partial)^2}\left[n_+\partial\mathfrak{h}_{\alpha_\perp\beta_\perp}n_+\partial\mathfrak{h}^{\alpha_\perp\beta_\perp} - n_+\partial\chi_c n_+\partial\chi_c\right]\right) + \mathcal{O}(\lambda^3), \tag{64}$$

which shows that $\mathfrak{h}$ counts as $\mathcal{O}(\lambda^2)$. The trace terms in $\mathcal{L}^{(2)}$ can therefore be dropped at this order, while the second interaction term in $\mathcal{L}^{(1)}$ is $\mathcal{O}(\lambda^2)$. These simplifications can be made manifest by expressing the Lagrangian in terms of $\mathfrak{h}_{\mu_\perp\nu_\perp}$ only.

这表明 $\mathfrak{h}$ 的阶数为 $\mathcal{O}(\lambda^2)$。因此 $\mathcal{L}^{(2)}$ 中的迹项在该阶可以舍去，而 $\mathcal{L}^{(1)}$ 中的第二个相互作用项是 $\mathcal{O}(\lambda^2)$ 阶。将拉格朗日量仅用 $\mathfrak{h}_{\mu_\perp\nu_\perp}$ 表示即可让这些简化变得明显。

The collinear Einstein-Hilbert Lagrangian for the graviton self-interactions and the one for the scalar field with φ^4 self-interactions are discussed in detail in [11]. Interestingly, the trilinear graviton self-interaction when expressed in terms of $\mathfrak{h}_{\mu_\perp\nu_\perp}$ (equivalently, light cone gauge) has a manifest double copy structure at the Lagrangian level [8].

引力子自相互作用的共线爱因斯坦-希尔伯特拉格朗日量，以及带 φ^4 自相互作用标量场的共线拉格朗日量已在文献 [11] 中详细讨论。有趣的是，当用 $\mathfrak{h}_{\mu_\perp\nu_\perp}$ 表示 (等效于光锥规范) 时，三引力子自相互作用在拉格朗日量层面具有明显的双重拷贝结构 [8]。

Soft-Collinear Gravity

软共线引力

The full soft-collinear EFT can be constructed separately for each collinear sector as seen from (41); therefore, the index i will be dropped. The first step is to introduce the soft and collinear modes. One way to implement the mode split consistently is to treat the collinear modes as small-distance fluctuations on top of the slowly varying soft background. For the metric field, the decomposition takes the form

如式 (41) 所示，完整的软共线有效场论可以对每个共线扇区分别构造，因此我们将省略指标 i 。第一步是引入软模和共线模。一致实现模式拆分的一种方式是，将共线模视为缓变软背景之上的小距离涨落。对于度规场，分解形式为

$$g_{\mu\upsilon} = \eta_{\mu\upsilon} + \kappa h_{\mu\upsilon} + \kappa s_{\mu\upsilon} \equiv g_{s\mu\upsilon} + \kappa h_{\mu\upsilon}, \tag{65}$$

where by definition all components of the soft field $s_{\mu\nu}$ have isotropic scaling

其中根据定义，软场 $s_{\mu\nu}$ 的所有分量都满足各向同性标度

$$s_{\mu\nu} \sim \lambda^2. \tag{66}$$

It is convenient to work with the soft background in terms of a soft metric tensor $g_{s\mu\nu}$ to leverage the geometric intuition. Therefore, this first step is fully equivalent to a standard weak-field expansion in $h_{\mu\nu}$ about a non-trivial dynamic background $g_{s\mu\nu}$.

借助软度规张量 $g_{s\mu\nu}$ 处理软背景十分方便，还能利用几何直观。因此这一步完全等价于在非平凡动态背景 $g_{s\mu\nu}$ 下对 $h_{\mu\nu}$ 做标准弱场展开。

The gauge symmetry is modified by this decomposition and changes into the gauge symmetry of the background (the soft gauge symmetry) and the gauge symmetry of the collinear fluctuations, which are covariant with respect to this background. The soft gauge symmetry is the standard diffeomorphism applied to the slowly varying background field. The transformations of the soft and collinear fluctuations read

该分解会修改规范对称性，使其变为背景的规范对称性 (软规范对称性) 和共线涨落的规范对称性，其中共线涨落相对于该背景是协变的。软规范对称性是作用在缓变背景场上的标准微分同胚。软涨落和共线涨落的变换形式为

$$h_{\mu\upsilon} \to \left[U_s\left(U_{s\mu}{}^{\alpha} U_{s\upsilon}{}^{\beta} h_{\alpha\beta}\right)\right], \tag{67}$$

$$\kappa s_{\mu\upsilon} \to \left[U_s\left(U_{s\mu}{}^{\alpha} U_{s\upsilon}{}^{\beta}\left(\eta_{\alpha\beta} + \kappa s_{\alpha\beta}\right)\right)\right] - \eta_{\mu\upsilon}.$$

It is understood that the x -dependence of $U_s(x)$ is the one of a soft field with variations over distances $\mathcal{O}\left(1/\lambda^2\right)$. The soft fluctuation $s_{\mu\nu}$ has the standard transformation of a metric perturbation, whereas the transformation of the collinear graviton $h_{\mu\nu}$ is the one of a standard rank-2 tensor, i.e., covariant like an ordinary matter field. Note that the soft metric tensor $g_{s\mu\nu}$ also transforms like a rank-2 tensor (13)

需要注意的是，$U_s(x)$ 对 x 的依赖符合软场的性质，仅在距离尺度 $\mathcal{O}(1/\lambda^2)$ 上发生变化。软涨落 $s_{\mu\nu}$ 满足度规微扰的标准变换，而共线引力子 $h_{\mu\nu}$ 的变换符合标准二阶张量的变换，即和普通物质场一样是协变的。注意软度规张量 $g_{s\mu\nu}$ 同样符合二阶张量的变换规则 (13)

$$g_{s\mu\nu} \to \left[U_s\left(U_{s\mu}{}^{\alpha} U_{s\nu}{}^{\beta} g_{s\alpha\beta}\right)\right]. \tag{68}$$

The interpretation of these transformations is clear: the background geometry is described by $g_{s\mu\nu}$, which transforms like an ordinary metric tensor, and $s_{\mu\nu}$ inherits a non-linear transformation from it. From the soft point of view, the collinear gravitons transform just like ordinary matter fields, i.e., like a standard rank-2 tensor, and not non-linearly like a metric perturbation. Therefore, from the soft perspective, any collinear field, regardless if gauge or matter, transforms like a matter field, i.e., according to its tensor representation.

这些变换的物理意义很明确: 背景几何由 $g_{s\mu\nu}$ 描述，它和普通度规张量变换规则相同，而 $s_{\mu\nu}$ 继承了它的非线性变换。从软的视角看，共线引力子和普通物质场的变换一致，即符合标准二阶张量的变换，而非像度规微扰那样做非线性变换。因此在软视角下，任何共线场，无论它是规范场还是物质场，都和物质场一样按自身的张量表示做变换。

The collinear gauge transformations are the transformations of the fields defined on the background metric and were already introduced in (32). They can be cast in the form

共线规范变换是定义在背景度规上的场变换，已经在式 (32) 中引入，可以写为如下形式

$$\kappa h_{\mu\nu} \to \left[U_c\left(U_{c\mu}{}^{\alpha} U_{c\nu}{}^{\beta} \kappa h_{\alpha\beta}\right)\right] + \left[U_c\left(U_{c\mu}{}^{\alpha} U_{c\nu}{}^{\beta} g_{s\alpha\beta}\right)\right] - g_{s\mu\nu}, \tag{69}$$

$$g_{s\mu\nu} \to g_{s\mu\nu},$$

which renders the transformation of $h_{\mu\nu}$ reminiscent of an ordinary gauge field transformation with a covariant and an inhomogeneous term. The fluctuation $h_{\mu\nu}$ has a non-linear transformation, similar to the standard weak-field expansion around flat space (20), but with respect to the soft background $g_{s\mu\nu}$, which appears in the inhomogeneous term instead of $\eta_{\mu\nu}$. Observe here that the soft background is necessarily invariant under the collinear transformations, since $U_c(x)$ is a collinear field.

这使得 $h_{\mu\nu}$ 的变换让人联想到普通规范场变换，包含协变项和非齐次项。涨落 $h_{\mu\nu}$ 做非线性变换，这和平直空间下的标准弱场展开 (20) 类似，但这里是相对于软背景 $g_{s\mu\nu}$ 的，软背景出现在非齐次项中，替代了原本 $\eta_{\mu\nu}$ 的位置。注意软背景在共线变换下必然不变，因为 $U_c(x)$ 是共线场。

Both gauge symmetries, soft and collinear, have the property that the full metric tensor $g_{\mu\nu} = g_{s\mu\nu} + \kappa h_{\mu\nu}$ transforms like the standard metric tensor. The transformations of matter fields follow the same logic. They can be found in [11].

软和共线这两种规范对称性都满足: 完整度规张量 $g_{\mu\nu} = g_{s\mu\nu} + \kappa h_{\mu\nu}$ 符合标准度规张量的变换规则。物质场的变换遵循相同逻辑，相关结果可见文献 [11]。

However, as already discussed in section "Light-Front Multipole Expansion," the transformations do not satisfy completely homogeneous λ -scaling, since they contain multiplications of collinear and soft fields at point x . The following construction identifies the background metric and the collinear field redefinition that renders the gauge symmetries compatible with a manifestly gauge-covariant form of the light-front multipole expansion.

但正如“光前多极展开”一节已经讨论过的，这些变换并不满足完全齐次的 λ 标度，因为其中包含点 x 处共线场与软场的乘积。接下来的构造会确定背景度规和共线场重新定义，使得规范对称性与明显规范协变形式的光前多极展开相容。

Riemann Normal Coordinates

黎曼法坐标

To gain some intuition for this procedure in the gravitational context, a simpler example is considered first: the static multipole expansion around the space-time point $x = 0$. The metric tensor in this theory should be multipole-expanded as

为了直观理解引力语境下的这一过程，我们先考虑一个更简单的例子: 时空点 $x = 0$ 周围的静态多极展开。该理论中的度规张量应展开为多极形式

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + x^\alpha \left[\partial_\alpha g_{\mu\nu}\right](0) + \frac{1}{2} x^\alpha x^\beta \left[\partial_\alpha \partial_\beta g_{\mu\nu}\right](0) + \mathcal{O}\left(x^3\right). \tag{70}$$

In gauge theories, the corresponding expansion of $A_\mu(x)$ is conveniently performed using Fock-Schwinger gauge $x^\mu A_\mu(x) = 0$. This gauge condition sets $A_\mu(0) = 0$, and the higher-order terms in the Taylor expansion are expressed in terms of the field strength tensor and its derivatives. Since the resulting expansion is manifestly gauge-covariant, the gauge fixing can easily be removed at the end. This technique has found numerous applications [34] to non-perturbative techniques in QCD, in which case short-distance fluctuations are treated in the background of the soft QCD vacuum.

在规范理论中，$A_\mu(x)$ 的对应展开通常通过福克-施温格规范 $x^\mu A_\mu(x) = 0$ 方便地完成。该规范条件规定了 $A_\mu(0) = 0$ ，泰勒展开中的高阶项可通过场强张量及其导数表示。由于得到的展开具有明显的规范协变性，因此可以很容易地在最后去掉规范固定。该技术已在 QCD 的非微扰方法中得到大量应用 [34]，在这类应用中，短距离涨落是在软 QCD 真空的背景下处理的。

In gravity, the analogous gauge corresponds to the familiar Riemann normal coordinates. The ”gauge condition” reads

在引力中，对应的规范就是我们熟知的黎曼法坐标。其“规范条件”为

$$x^\mu x^\nu \Gamma^\alpha_{\mu\nu}(x) = 0. \tag{71}$$

Compared to Fock-Schwinger gauge $x^\mu A_\mu(x) = 0$, one immediately notices that in gravity, the gauge condition starts at second order in x and restricts the derivative of the metric tensor at $x = 0$, but not $g_{\mu\nu}(0)$, since $\Gamma \sim \partial g$. Indeed, the effect of Riemann normal coordinates is to set the first derivative $\left[\partial_\alpha g_{\mu\nu}\right](0) = 0$

and to express the higher derivatives via the Riemann tensor and its derivatives at $x=0$. The metric tensor near $x=0$ is then expressed as

与福克-施温格规范 $x^\mu A_\mu(x)=0$ 相比，我们可以立刻发现，在引力中规范条件从 x 的二阶开始，它限制的是 $x=0$ 处度规张量的导数，而非 $g_{\mu\nu}(0)$ ，原因是 $\Gamma\sim\partial g$ 。事实上，黎曼法坐标的作用就是令一阶导数 $\left[\partial_\alpha g_{\mu\nu}\right](0)=0$ 为零，再通过 $x=0$ 处的黎曼张量及其导数表示更高阶导数。于是 $x=0$ 附近的度规张量可以写为

$$\tilde{g}_{\mu\nu}(x)=g_{\mu\nu}(0)-\frac{1}{3}x^\alpha x^\beta R_{\alpha\mu\beta\nu}(0)+\mathcal{O}\left(x^3\right). \tag{72}$$

The gauge condition (71) does not fix the diffeomorphism symmetry completely, but only up to global linear transformations $A_\mu{}^\alpha\in\mathrm{GL}(1,3)$,

规范条件 (71) 并没有完全固定微分同胚对称性，仅将其限制到整体线性变换 $A_\mu{}^\alpha\in\mathrm{GL}(1,3)$ ，

$$x^\mu\to A_\alpha{}^\mu x^\alpha \tag{73}$$

Using this residual symmetry, one can transform the leading term $g_{\mu\nu}(0)$ to the Minkowski metric $\eta_{\mu\nu}$, and the metric tensor in these coordinates reads

利用这一剩余对称性，我们可以将领头项 $g_{\mu\nu}(0)$ 变换为闵氏度规 $\eta_{\mu\nu}$ ，此时该坐标系下的度规张量为

$$\tilde{g}_{\mu\nu}(x)=\eta_{\mu\nu}-\frac{1}{3}x^\alpha x^\beta R_{\alpha\mu\beta\nu}(0)+\mathcal{O}\left(x^3\right), \tag{74}$$

which is the standard form of the metric in Riemann normal coordinates. The remaining residual symmetries of the full diffeomorphisms are now the ones of the Minkowski metric $\eta_{\mu\nu}$, i.e., global Lorentz transformations.

这就是黎曼法坐标中度规的标准形式。此时完整微分同胚剩余的对称性就是闵氏度规 $\eta_{\mu\nu}$ 的对称性，即整体洛伦兹变换。

The lesson of this section is the following: using Riemann normal coordinates, one can express the multipole expansion of the metric tensor in terms of the metric at the origin and Riemann tensor terms. Then, one can exploit the residual symmetry to diagonalize this metric and change it to the flat-space one at $x=0$. The result is a covariant multipole expansion that features global Poincaré transformations as residual symmetries.

本节的结论如下: 利用黎曼法坐标，我们可以将度规张量的多极展开用原点处的度规和黎曼张量项表示。随后，可以利用剩余对称性将该度规对角化，在 $x=0$ 处将其变为平直空间度规。最终得到的是一个协变多极展开，其剩余对称性为整体庞加莱变换。

Fixed-Line Normal Coordinates

定线法线坐标

The above discussion motivates a generalization of the Riemann normal coordinates adapted to the situation where the physical system is not localized around a spacetime point, but around the light-like trajectory $x_-^\mu = n_+ x \frac{n_-^\mu}{2}$, which allows for the light-front multipole expansion of soft fluctuations. The metric tensor is now expanded as

上述讨论引出了黎曼法线坐标的一种推广，它适配物理系统不局域在时空点附近、而是局域在类光轨迹 $x_-^\mu = n_+ x \frac{n_-^\mu}{2}$ 周围的情况，可用于软涨落的光前多极展开。度规张量此时展开为

$$g_{s\mu\nu}(x) = g_{s\mu\nu}(x_-) + x_\perp^\alpha \left[\partial_\alpha g_{s\mu\nu}\right](x_-) + \frac{1}{2} n_- x \left[n_+ \partial g_{s\mu\nu}\right](x_-)$$

$$+\frac{1}{2} x_\perp^\alpha x_\perp^\beta \left[\partial_\alpha \partial_\beta g_{s\mu\nu}\right](x_-) + \mathcal{O}\left(\lambda^3 g_{s\mu\nu}\right), \tag{75}$$

which differs from the previous one (70) in two aspects: first, the expansion is only in the directions orthogonal to the light-like line x_-^μ , and second, the coefficients in the Taylor expansion are dynamical fields and depend on x_-^μ . In gauge theory, the generalization of Fock-Schwinger gauge is "fixed-line" gauge [5] $(x - x_-)^\mu A_\mu(x_-) = 0$. In gravity, this suggests the fixed-line normal coordinate condition [11]

它和之前的式 (70) 有两点不同: 第一, 展开仅在垂直于类光线 x_-^μ 的方向进行; 第二, 泰勒展开的系数是动力学场,依赖于 x_-^μ 。规范理论中,福克-施温格规范的推广就是“定线”规范 [5] $(x - x_-)^\mu A_\mu(x_-) = 0$ 。在引力中，这给出了定线法线坐标条件 [11]

$$(x - x_-)^\alpha (x - x_-)^\beta \Gamma^\mu_{\alpha\beta}(x) = 0, \tag{76}$$

which replaces (71). Suppose that $\Gamma^\mu_{\alpha\beta}(x)$ does not satisfy this condition; then one can find new coordinates $\tilde{x}^\mu$, related to x^μ by

该条件替代了式 (71)。假设 $\Gamma^\mu_{\alpha\beta}(x)$ 不满足该条件，那么我们可以找到新坐标 $\tilde{x}^\mu$ ，它和 x^μ 满足如下关系

$$\tilde{x}^\mu = x^\mu + \frac{1}{2}(x - x_-)^\alpha (x - x_-)^\beta \Gamma^\mu_{\alpha\beta} \tag{77}$$

$$+\frac{1}{6}(x - x_-)^\alpha (x - x_-)^\beta (x - x_-)^\nu \left(\Gamma^\mu_{\alpha\tau}\Gamma^\tau_{\beta\nu} + \left[\partial_\nu \Gamma^\mu_{\alpha\beta}\right]\right)$$

$$+\mathcal{O}\left((x - x_-)^4\right)$$

such that $\tilde{\Gamma}^\mu_{\alpha\beta}(x)$ fulfils (76). In the above and for the remainder of section "Soft-Collinear Gravity," the following convention is adopted: whenever a soft field appears without explicit position argument, the field is evaluated at x_-^μ .

使得 $\tilde{\Gamma}^\mu_{\alpha\beta}(x)$ 满足式 (76)。在上述讨论以及“软共线引力”章节余下内容中，采用以下约定: 凡是软场未显式给出位置参数，就表示该场在 x_-^μ 处取值。

Once again, the fixed-line normal coordinate gauge condition (76) does not fix the gauge completely. For example,(76) leaves the components $\tilde{\Gamma}^\mu_{--}(\tilde{x})$ unconstrained, and (77) does not affect $g_{s\mu\nu}(x_-)$, since it is second order in $(x-x_-)^\mu$. The remaining diffeomorphisms form a residual symmetry, which is the analogue of the global symmetry transformations from before (73), but since the parameters depend on x_-^μ , it is a gauge symmetry. As for the Riemann normal coordinates, the residual transformations can be used to further simplify the residual metric tensor $g_{s\mu\nu}(x_-)$ by performing a linear coordinate transformation locally at x_-^μ in the components orthogonal to x_-^μ . After this step, one can identify the homogeneous soft background field. To this end, introduce the vierbein

定线法线坐标规范条件 (76) 同样没有完全固定规范。例如，式 (76) 未约束分量 $\Gamma^\mu_{--}(\bar{x})$ ，而式 (77) 不影响 $g_{s\mu\nu}(x_-)$ ，因为它是 $(x-x_-)^\mu$ 二阶小量。剩余微分同胚构成残差对称性，它对应之前式 (73) 的整体对称性变换，但由于参数依赖 x_-^μ ，它是一种规范对称性。和黎曼法线坐标的情况类似，可以利用残差变换在 x_-^μ 处对垂直于 x_-^μ 的分量做局部线性坐标变换，进一步简化残差度规张量 $g_{s\mu\nu}(x_-)$ 。这一步之后可以得到齐次软背景场。为此，引入标架

$$g_{s\mu\nu}(x_-) \equiv e_\mu{}^\alpha(x_-)\, e_\nu{}^\beta(x_-)\, \eta_{\alpha\beta} \tag{78}$$

and its inverse

及其逆

$$E^\mu_\alpha(x_-)\, e^\alpha_\nu(x_-) = \delta^\mu_\nu. \tag{79}$$

These objects can be thought of as matrices that locally diagonalize the metric tensor to express it in terms of the flat-space metric $\eta_{\mu\nu}$. Performing the linear transformation $\bar{x}^\mu \to \check{x}^\mu(\bar{x})$ defined by

可以将这些量视为矩阵，它们将度规局部对角化，用平直空间度规 $\eta_{\mu\nu}$ 表示原度规。对由下式定义的线性变换 $\bar{x}^\mu \to \check{x}^\mu(\bar{x})$

$$n_+\check{x} = n_+\bar{x},\ \check{x}^\mu_\perp = e_\alpha{}^{\mu_\perp}\bar{x}^\alpha,\ n_-\check{x} = n_{-\rho}e_\alpha{}^\rho\bar{x}^\alpha, \tag{80}$$

which leaves x_-^μ invariant, yields a new coordinate system $\check{x}$, the fixed-line normal coordinates (FLNC). The original coordinate can be expressed in terms of the FLNC $\check{x}^\mu$ as

该变换保持 x_-^μ 不变，得到新坐标系 $\check{x}$ ，即定线法线坐标 (FLNC)。原坐标可以用定线法线坐标 $\check{x}^\mu$ 表示为

$$x^\mu = \check{x}^\mu + \theta^\mu_{\text{FLNC}}(\check{x}), \tag{81}$$

with parameter

其中参数为

$$\theta^\mu_{\text{FLNC}}(x) = \left(E^\mu{}_\rho - \delta^\mu_\rho\right)(x-x_-)^\rho - \frac{1}{2}(x-x_-)^\rho(x-x_-)^\sigma E^\alpha{}_\rho E^\beta{}_\sigma \Gamma^\mu{}_{\alpha\beta}$$

$$+\frac{1}{6}(x-x_-)^\rho(x-x_-)^\sigma(x-x_-)^\lambda E^\alpha{}_\rho E^\beta{}_\sigma E^\nu{}_\lambda$$

$$\times \left(2\Gamma^{\mu}_{\alpha\tau}\Gamma^{\tau}_{\beta\nu} - \left[\partial_{\nu}\Gamma^{\mu}_{\alpha\beta}\right]\right) + \mathcal{O}\left(x^4\right). \tag{82}$$

Note that every $(x - x_{-})^{\mu}$ generates a contraction with the inverse vierbein as a result of this linear transformation. Returning now to the active point of view, the translation operator $T_{\theta_{\mathrm{FLNC}}}(x)$ corresponding to the coordinate change to $\breve{x}^{\mu}$ defines a new "Wilson line" $R_{\mathrm{FLNC}}(x)$ via

注意，该线性变换会让每个 $(x - x_{-})^{\mu}$ 和逆标架缩并。现在回到主动视角，变换到坐标 $\breve{x}^{\mu}$ 对应的平移算符 $T_{\theta_{\mathrm{FLNC}}}(x)$ 通过下式定义了新的“威尔逊线” $R_{\mathrm{FLNC}}(x)$

$$R^{-1}_{\mathrm{FLNC}}(x) = T_{\theta_{\mathrm{FLNC}}}(x). \tag{83}$$

As an intermediate result, one can compute the metric tensor in fixed-line normal coordinates, denoted by $\breve{g}_{s\mu\nu}$, using R_{FLNC} as

作为中间结果，我们可以在定线法线坐标下计算度规张量，记为 $\breve{g}_{s\mu\nu}$ ，利用 R_{FLNC} 得到

$$\breve{g}_{s\mu\nu}(x) \equiv R^{\alpha}_{\ \mu}(x) R^{\beta}_{\ \nu}(x) \left[R^{-1}_{\mathrm{FLNC}}(x)\, g_{s\alpha\beta}(x)\right], \tag{84}$$

with

其中

$$R^{\alpha}_{\ \mu}(x) = \frac{\partial x^{\alpha}}{\partial \breve{x}^{\mu}}(x) \tag{85}$$

the Jacobian of the transformation. As was the case for Riemann normal coordinates, the metric in FLNC can now be split into a background metric (simply $\eta_{\mu\nu}$ for the former), which forms the leading term, and the multipole series of manifestly covariant terms expressed via the Riemann tensor. Therefore, it is convenient to split the metric tensor as

该变换的雅可比行列式。与黎曼法坐标的情况相同，固定线法坐标 (FLNC) 中的度规可拆分为主导项背景度规 (黎曼法坐标中就是 $\eta_{\mu\nu}$)，以及由黎曼张量表示的明显协变项的多极级数。因此我们可以方便地将度规张量拆分为

$$\breve{g}_{s\mu\nu}(x) \equiv \hat{g}_{s\mu\nu}(x) + \mathfrak{g}_{s\mu\nu}(x), \tag{86}$$

where $\mathfrak{g}_{s\mu\nu}$ are the manifestly gauge-covariant terms, i.e., the Riemann tensor terms, and the background field is $\hat{g}_{s\mu\nu}(x)$. It is given by

其中 $\mathfrak{g}_{s\mu\nu}$ 是明显规范协变的项，即黎曼张量项，背景场为 $\hat{g}_{s\mu\nu}(x)$ 。其表达式为

$$\hat{g}_{s+-}(x) = e_{-+} - (x - x_{-})^{\alpha}[\Omega_{-}]_{\alpha+}, \tag{87}$$

$$\hat{g}_{s\mu_{\perp}-}(x) = e_{-\mu_{\perp}} - (x - x_{-})^{\alpha}[\Omega_{-}]_{\alpha\mu_{\perp}}, \tag{88}$$

$$\hat{g}_{s--}(x) = \left(e_-{}^{\alpha} - (x-x_-)^{\rho}[\Omega_-]_{\rho}{}^{\alpha}\right)\left(e_-{}^{\beta} - (x-x_-)^{\sigma}[\Omega_-]_{\sigma}{}^{\beta}\right)\eta_{\alpha\beta},$$

(89)

$$\hat{g}_{s\mu_\perp\nu_\perp}(x) = \eta_{\mu_\perp\nu_\perp}, \tag{90}$$

$$\hat{g}_{s+\perp}(x) = \hat{g}_{s++}(x) = 0. \tag{91}$$

Here, the fields on the right-hand side implicitly depend on x_-, i.e., $e_{-+} \equiv e_{-+}(x_-)$, and $[\Omega_-]_{\alpha\beta}$ is the spin connection, defined as [7]

此处，右侧的场隐含依赖于 x_-，也就是 $e_{-+} \equiv e_{-+}(x_-)$，$[\Omega_-]_{\alpha\beta}$ 是自旋联络，定义为 [7]

$$[\Omega_-]^{\alpha\beta} = e^{\alpha}_{\upsilon}\left[\partial_- E^{\upsilon\beta}\right] + e^{\alpha}_{\upsilon}\Gamma^{\upsilon}_{\sigma-}E^{\sigma\beta}. \tag{93}$$

---

[7] The definitions (78) and (79) can be extended from the light cone x^{μ}_{-} to x^{μ}, in which case

[7] 定义 (78) 和 (79) 可以从光锥 x^{μ}_{-} 推广到 x^{μ}，推广后

$$\left[\Omega_{\mu}\right]^{\alpha\beta}(x) = e^{\alpha}_{\upsilon}\left[\partial_{\mu}E^{\upsilon\beta}\right](x) + e^{\alpha}_{\upsilon}\Gamma^{\upsilon}_{\sigma\mu}E^{\sigma\beta}(x) \tag{92}$$

coincides with the usual spin connection. The background field construction only needs $[\Omega_-]^{\alpha\beta}(x_-)$, which is well defined on the light cone, since only the derivative $n_-\partial$ along the light cone appears in (93).

与通常的自旋联络一致。背景场构造仅需要 $[\Omega_-]^{\alpha\beta}(x_-)$，它在光锥上是良好定义的，因为式 (93) 中仅出现沿光锥的导数 $n_-\partial$。

Note that the components orthogonal to x^{μ}_{-} coincide with the flat-space metric, as in these directions the light-front multipole expansion is the same as the static one.

注意，与 x^{μ}_{-} 正交的分量与平直空间度规一致，因为在这些方向上，光前多极展开与静态多极展开是相同的。

The remainder term $\mathfrak{g}_{s\mu\upsilon}$ in (86) is manifestly gauge-covariant and depends only on the Riemann tensor. Up to $\mathcal{O}\left(\lambda^4\right)$, it is given by

式 (86) 中的余项 $\mathfrak{g}_{s\mu\upsilon}$ 是明显规范协变的，且仅依赖于黎曼张量。到 $\mathcal{O}\left(\lambda^4\right)$ 阶，它的表达式为

$$\mathfrak{g}_{s\mu\upsilon}(x) = -\frac{n_{+\mu}n_{+\upsilon}}{4}x^{\alpha}_{\perp}x^{\beta}_{\perp}R_{\alpha-\beta-} - \frac{n_{+\mu}}{2}\frac{2}{3}x^{\alpha}_{\perp}x^{\beta}_{\perp}R_{\alpha\upsilon_\perp\beta-} - \frac{n_{+\upsilon}}{2}\frac{2}{3}x^{\alpha}_{\perp}x^{\beta}_{\perp}R_{\alpha\mu_\perp\beta-}$$

$$
-\left(\frac{n_{+\mu}n_{-\upsilon}}{4}+\frac{n_{+\upsilon}n_{-\mu}}{4}\right)\frac{2}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha+\beta-}-\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\mu_\perp\beta\upsilon_\perp} \tag{94}
$$

$$
-\frac{n_{-\mu}}{2}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\upsilon_\perp\beta+}-\frac{n_{-\upsilon}}{2}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\mu_\perp\beta+}-\frac{n_{-\mu}n_{-\upsilon}}{4}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha+\beta+}.
$$

The explicit expressions of $\hat{g}_{s\mu\nu}$ and $\mathfrak{g}_{s\mu\nu}$ obtained from this construction are important results, while the intermediate $\breve{g}_{s\mu\nu}$ is no longer employed in the following.

通过该构造得到的 $\hat{g}_{s\mu\nu}$ 和 $\mathfrak{g}_{s\mu\nu}$ 的显式表达式是重要结果，下文将不再使用中间量 $\breve{g}_{s\mu\nu}$ 。

The result for $\hat{g}_{s\mu\nu}$ contains two gauge fields appearing only in the n_-^μ components: the vierbein and the spin connection. They can be expanded in terms of the soft graviton field as

$\hat{g}_{s\mu\nu}$ 的结果仅包含两个出现在 n_-^μ 分量中的规范场: vierbein(标架场) 和自旋联络。它们可以按软引力子场展开为

$$
e_-{}^\alpha=\delta_-^\alpha+\frac{\kappa}{2}s_-^\alpha-\frac{\kappa^2}{8}s_{-\beta}s^{\beta\alpha}+\mathcal{O}\left(s^3\right), \tag{95}
$$

$$
[\Omega_-]_{\alpha\beta}=-\frac{\kappa}{2}\left(\left[\partial_\alpha s_{\beta-}\right]-\left[\partial_\beta s_{\alpha-}\right]\right)+\mathcal{O}\left(s^2\right). \tag{96}
$$

Note that the spin connection is defined as the derivative of the metric, resp. vierbein. In the EFT, the soft metric and vierbein are always evaluated at x_-^μ . Thus, in the EFT, $\partial_\perp s_{\mu\upsilon}(x_-)=0$ if the derivative were taken after setting $x=x_-$. This means that the vierbein and the spin connection above should be interpreted as two truly independent gauge fields from the EFT perspective, since the derivative contains additional information that cannot be obtained from the vierbein in the EFT.

注意，自旋联络定义为度规 (对应标架场) 的导数。在有效场论 (EFT) 中，软度规和标架场始终在 x_-^μ 处取值。因此，若我们先取定 $x=x_-$ 再求导，在 EFT 中就会得到 $\partial_\perp s_{\mu\upsilon}(x_-)=0$ 。这意味着从 EFT 的视角来看，上述标架场和自旋联络应当被解释为两个真正独立的规范场，因为导数包含了 EFT 中无法从标架场得到的额外信息。

Corresponding to these two independent gauge fields, the soft gauge transformations induce two emergent soft gauge symmetries that take the form of local translations and local Lorentz transformations on the light cone x_-^μ of the collinear modes and global transformations on the hyper-planes of fixed x_-^μ . More precisely, performing a gauge transformation $U_s(x)=T_\varepsilon^{-1}$ with local translation parameter $\varepsilon_\mu(x)$, $\hat{g}_{s\mu\nu}$ does not transform with T_ε^{-1} , but instead with $T_{\varepsilon+\omega}^{-1}$, defined by the parameters $\varepsilon_\mu(x_-)$ and $\omega_{\mu\upsilon}(x_-)=-\frac{1}{2}\left(\left[\partial_\mu\varepsilon_\upsilon\right]-\left[\partial_\upsilon\varepsilon_\mu\right]\right)(x_-)$, which again should be regarded as two independent transformations. The corresponding infinitesimal coordinate transformation is given by

对应这两个独立规范场，软规范变换诱导出两个涌现软规范对称性，其形式为: 对共线模式的光锥 x_-^μ 取局域平移变换和局域洛伦兹变换，对固定 x_-^μ 超平面取整体变换。更准确地说，用局域平移参数 $\varepsilon_\mu(x)$, $\hat{g}_{s\mu\nu}$ 做规范变换 $U_s(x)=T_\varepsilon^{-1}$ 时，变换并非由 T_ε^{-1} 给出，而是由 $T_{\varepsilon+\omega}^{-1}$ 给出，$T_{\varepsilon+\omega}^{-1}$ 由参数 $\varepsilon_\mu(x_-)$ 和 $\omega_{\mu\upsilon}(x_-)=-\frac{1}{2}\left(\left[\partial_\mu\varepsilon_\upsilon\right]-\left[\partial_\upsilon\varepsilon_\mu\right]\right)(x_-)$ 定义，这二者应视为两个独立变换。对应的无穷小坐标变换为

$$x^{\mu} \to x^{\mu} + \kappa\varepsilon^{\mu}(x_-) + \kappa\omega^{\mu}_{\ \upsilon}(x_-)(x - x_-)^{\upsilon}. \tag{97}$$

The first term stems from the transformation of the vierbein, while the second one, proportional to $(x - x_-)^{\mu}$, arises from the spin connection transformation. The two gauge symmetries of the collinear fields in the effective theory are emergent, since they have a different physical meaning than the original one. The emergent soft gauge symmetries, and consequently the soft fields, live only along the classical trajectory of the collinear particles x_-^{μ} and are "visible" only to the collinear fields in direction i. Other collinear directions see their own soft gauge symmetry. The symmetries of the different collinear sectors are induced by the gauge symmetry of the full theory and connected through the purely soft part of the Lagrangian, which continues to transform under the full soft symmetry parameter $\varepsilon_{\mu}(x)$.

第一项来源于 Vierbein 的变换，而第二项正比于 $(x - x_-)^{\mu}$，来源于自旋联络变换。有效理论中共线场的这两个规范对称性是涌现的，因为它们与原规范对称性的物理含义不同。涌现软规范对称性以及由此对应的软场仅存在于共线粒子 x_-^{μ} 的经典轨迹上，且仅对方向 i 的共线场"可见"。其他共线方向拥有各自的软规范对称性。不同共线区域的对称性由完整理论的规范对称性诱导，并通过拉格朗日的纯软部分连接，该部分始终在完整软对称参数 $\varepsilon_{\mu}(x)$ 下变换。

The effective soft gauge fields $e_-{}^{\alpha}(x_-)$ and $[\Omega_-]_{\alpha\beta}(x_-)$ are non-linear functions of $s_{\mu\nu}$. Since the residual background metric has the property that only $\hat{g}_{s\mu-}$ is nonvanishing, it is possible to introduce a soft covariant derivative. To this end, we define $\widehat{E}_s{}^{\mu}{}_a$ through $\hat{g}_s^{\mu\upsilon} = \eta^{ab}\widehat{E}_s{}^{\mu}{}_a\widehat{E}_s{}^{\upsilon}{}_b$, such that $\widehat{E}_s{}^{\mu}{}_a$ can be obtained from inverting (87)-(91), i.e., $\widehat{E}_s{}^{\mu}{}_a$ is the inverse of

有效软规范场 $e_-{}^{\alpha}(x_-)$ 和 $[\Omega_-]_{\alpha\beta}(x_-)$ 是 $s_{\mu\nu}$ 的非线性函数。由于剩余背景度规满足仅 $\hat{g}_{s\mu-}$ 非零的性质，我们可以引入软协变导数。为此，我们通过 $\hat{g}_s^{\mu\upsilon} = \eta^{ab}\widehat{E}_s{}^{\mu}{}_a\widehat{E}_s{}^{\upsilon}{}_b$ 定义 $\widehat{E}_s{}^{\mu}{}_a$，因此可以通过对式 (87)-(91) 求逆得到 $\widehat{E}_s{}^{\mu}{}_a$，即 $\widehat{E}_s{}^{\mu}{}_a$ 是下式的逆

$$\hat{e}_s{}^a{}_{\mu}(x) = \left(\delta^a_{\perp\mu} + \frac{n_{-\mu}}{2}n_+^a + \frac{n_{+\mu}}{2}\left(e_s{}^a{}_-(x_-) + y^{\beta}[\Omega_-]^a{}_{\beta}(x_-)\right)\right), \tag{98}$$

and only $\widehat{E}_{s-}^{\mu}$ differs from δ_a^{μ}. The soft covariant derivative is then given by

且仅有 $\widehat{E}_{s-}^{\mu}$ 与 δ_a^{μ} 不同。软协变导数则由下式给出

$$n_-D_s \equiv \widehat{E}^{\mu}_{s-}\partial_{\mu} \tag{99}$$

$$= \partial_- - \frac{\kappa}{2}s_{-\mu}\partial^{\mu} + \frac{\kappa^2}{8}s_{+-}s_{--}n_+\partial + \frac{\kappa^2}{16}s_{-\alpha_\perp}s_-^{\alpha_\perp}n_+\partial + \frac{\kappa}{2}[\Omega_-]_{\mu\nu}J^{\mu\nu} + \mathcal{O}(\lambda^3),$$

where

其中

$$J^{\mu\upsilon} = (x - x_-)^{\mu}\partial^{\upsilon} - (x - x_-)^{\upsilon}\partial^{\mu} \tag{100}$$

is the angular momentum (Lorentz generator) operator. The form of (99) is significant and revealing. Usually, a scalar field coupled to gravity does not have a covariant derivative. However, in SCET gravity, the residual symmetry corresponds to local Lorentz transformations that live only on the classical trajectory of the energetic particles. A scalar field has a non-trivial transformation under this symmetry and therefore requires the introduction of a covariant derivative. Namely, if a collinear scalar field transforms under (97) infinitesimally as

是角动量 (洛伦兹生成元) 算符。式 (99) 的形式意义重大且具有启发性。通常，耦合引力的标量场不需要协变导数。然而在 SCET 引力中，剩余对称性对应仅存在于高能粒子经典轨迹上的局域洛伦兹变换。标量场在该对称性下存在非平凡变换，因此需要引入协变导数。也就是说，如果共线标量场在式 (97) 下的无穷小变换为

$$T_{\varepsilon+\omega}^{-1}\varphi_c(x) = 1 - \kappa\varepsilon^{\alpha}\partial_{\alpha}\varphi_c(x) - \kappa\omega_{\alpha\beta}(x-x_-)^{\beta}\partial^{\alpha}\varphi_c(x) + \mathcal{O}\left(\varepsilon^2\right), \tag{101}$$

one can check that the derivative (99) has the covariant transformation

可以验证导数式 (99) 满足协变变换

$$n_-D_s\varphi_c(x) \to T_{\varepsilon+\omega}^{-1}\left[n_-D_s\varphi_c(x)\right] - \kappa\omega_{-\alpha}D_s^{\alpha}\varphi_c(x) + \mathcal{O}\left(\varepsilon^2\right). \tag{102}$$

The transverse partial derivatives $\partial_{\perp}$ and $n_+\partial$ already transform as in (102), and no modification is necessary.

横向偏导数 $\partial_{\perp}$ 和 $n_+\partial$ 已经按照式 (102) 变换，无需进行任何修改。

The soft covariant derivative (99) contains the two independent gauge fields. At the linear order in $s_{\mu\nu}$, corresponding to the minimal coupling terms in the Lagrangian, the first gauge field $e_{-\mu}$ couples to the momentum $P^{\mu} \equiv -i\partial^{\mu}$, while the second gauge field, the spin connection, couples to the angular momentum of the scalar field. For representations with non-vanishing spin, the total angular momentum would appear in (99) [9]. In the case of several directions, the object $n_{i-}D_s$ appears inside the i -collinear Lagrangian $\mathcal{L}_{c_i}$. In this sense, one has $i = 1, \ldots, N$ gauge symmetries that are restricted to their respective collinear trajectories, all ultimately tied to the same background soft field through the purely soft Lagrangian $\mathcal{L}_s\left[s_{\mu\nu}(x)\right]$ in (41).

软协变导数 (99) 包含两个独立规范场。在 $s_{\mu\nu}$ 的一阶线性项下，对应拉格朗日量中的最小耦合项，第一个规范场 $e_{-\mu}$ 耦合到动量 $P^{\mu} \equiv -i\partial^{\mu}$ ，而第二个规范场即自旋联络耦合到标量场的角动量。对于自旋非零的表示，总角动量会出现在式 (99) 中 [9]。当存在多个共线方向时，量 $n_{i-}D_s$ 出现在 i 共线拉格朗日量 $\mathcal{L}_{c_i}$ 内部。从这个意义上说，存在 $i = 1, \ldots, N$ 个限制在各自共线轨迹上的规范对称性，所有这些对称性最终都通过式 (41) 中的纯软拉格朗日量 $\mathcal{L}_s\left[s_{\mu\nu}(x)\right]$ 关联到同一个背景软场。

It is important to stress that, as seen in (99) and (94), the fields $\hat{g}_{s\mu\nu}$ and $\mathfrak{g}_{s\mu\nu}$ are expressed in terms of the original soft graviton field $s_{\mu\nu}$, where no gauge is fixed. This is where the residual gauge symmetry stems from. The FLNC coordinates are merely a tool to construct the covariant light-front multipole expansion of the EFT, and one does not fix this gauge in the final Lagrangian, which is a function of $s_{\mu\nu}$. Similar to the case of light cone gauge in the collinear sector, using the analogue of Wilson lines, the FLNC condition can be implemented ”covariantly” by redefining the collinear fields, without restricting the gauge freedom of the soft field.

需要着重强调的是，正如式 (99) 和 (94) 所示，场 $\hat{g}_{s\mu\nu}$ 和 $\mathfrak{g}_{s\mu\nu}$ 由原始未固定规范的软引力子场 $s_{\mu\nu}$ 表示，剩余规范对称性正来源于此。定线法坐标 (FLNC) 只是构造有效场论协变光前多极展开的工具，我们并未在最终拉格朗日量中固定该规范，最终拉格朗日量是 $s_{\mu\nu}$ 的函数。与共线 sector 中光锥规范的情况类似，利用威尔逊线的类似构造，FLNC 条件可以通过重新定义共场“协变地”实现，而不会限制软场的规范自由度。

Soft-Collinear Lagrangian

软共线拉格朗日量

Equipped with these concepts, it is not difficult to derive the effective Lagrangian by purely algebraic manipulations. As a first step, one redefines the collinear fields using the collinear ”Wilson line” W_c (50) and the previously defined R -”Wilson line” (83)[8] to obtain collinear fields that have gauge transformations compatible with the multipole expansion. These new fields are denoted by $\hat{\varphi}_c$ and $\widehat{h}_{\mu\nu}$ and are given by [9]

有了这些概念，通过纯代数操作推导有效拉格朗日量并不困难。第一步，我们利用共线“威尔逊线” W_c (50) 和之前定义的 R -“威尔逊线”(83)[8] 重新定义共线场，得到规范变换与多极展开兼容的共线场。这些新场记为 $\hat{\varphi}_c$ 和 $\widehat{h}_{\mu\nu}$ ，形式为 [9]

$$\varphi_c = \left[R_{\rm FLNC} W_c^{-1} \hat{\varphi}_c\right] = \left[R_{\rm FLNC} \hat{\chi}_c\right], \tag{103}$$

$$\begin{aligned} \kappa h_{\mu\nu} &= \left[R_{\rm FLNC} R_\mu{}^\alpha R_\nu{}^\beta \left(W^\rho{}_\alpha W^\sigma{}_\beta \left[W_c^{-1}\left(\kappa \widehat{h}_{\rho\sigma} + \hat{g}_{s\rho\sigma}(x)\right)\right] - \hat{g}_{s\alpha\beta}(x)\right)\right] \\ &= \left[R_{\rm FLNC} R_\mu{}^\alpha R_\nu{}^\beta \hat{\mathfrak{h}}_{\alpha\beta}\right]. \end{aligned} \tag{104}$$

Here, the collinear fields on the left-hand side are assumed to be in light cone gauge, but the gauge is not fixed on the right-hand side. The right-hand sides also identify the collinear gauge-invariant building blocks $\hat{\chi}_c$ and $\hat{\eta}_{\mu\nu}$ after the field redefinition, i.e., they are invariant with respect to the compatible gauge symmetries.

此处，左侧的共线场默认处于光锥规范下，但右侧不固定规范。场重定义后，右侧也给出了共线规范不变构造块 $\hat{\chi}_c$ 和 $\hat{\eta}_{\mu\nu}$ ，即它们相对于相容的规范对称性是不变的。

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[8] In the following, $R_{\rm FLNC}$, its Jacobians $R_\mu{}^\alpha$, and its determinant $\det\left(\underline{R}\right)$ are understood to be evaluated at x if no argument is given.

[8] 在下文中，若无特别说明参数，$R_{\rm FLNC}$ 、其雅可比矩阵 $R_\mu{}^\alpha$ 及其行列式 $\det\left(\underline{R}\right)$ 均默认在 x 处取值。

[9] There is one subtlety: the collinear ”Wilson line” in the soft-collinear theory is different from the purely collinear one at the non-linear level, as the soft background field has a non-vanishing $\hat{g}_{s+-}$ component, and therefore appears in the transformation of $\widehat{h}_{+-}$. This is explained in [11].

[9] 有一点需要注意: 软共线理论中的共线“威尔逊线”在非线性层面上与纯共线情形不同，因为软背景场存在非零的 $\hat{g}_{s+-}$ 分量，因此会出现在 $\hat{h}_{+-}$ 的变换中。文献 [11] 对此有解释。

To illustrate the calculation of the effective Lagrangian, consider the collinear matter field in the soft background $g_{s\mu\nu}(x)$ as defined in (65), [10]

为了说明有效拉格朗日量的计算，我们考虑式 (65) 定义的软背景 $g_{s\mu\nu}(x)$ 中的共线物质场，[10]

$$\mathcal{L}_\varphi = \frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}\partial_\mu\varphi_c\partial_\nu\varphi_c. \tag{105}$$

Inserting (103) results in

代入 (103) 后得到

$$\begin{aligned}\mathcal{L}_\varphi &= \frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}\left[\partial_\mu\left(R_{\mathrm{FLNC}}\hat{\chi}_c\right)\right]\left[\partial_\nu\left(R_{\mathrm{FLNC}}\hat{\chi}_c\right)\right] \\ &= \frac{1}{2}\det\left(\underline{R}\right)\left[R_{\mathrm{FLNC}}^{-1}\sqrt{-g_s}\right]R_\mu{}^\alpha R_\nu{}^\beta\left[R_{\mathrm{FLNC}}^{-1}g_s^{\mu\nu}\right]\partial_\alpha\hat{\chi}_c\partial_\beta\hat{\chi}_c.\end{aligned} \tag{106}$$

The metric $\breve{g}^{\alpha\beta}(x)$ from (84) can now be identified, together with the corresponding metric determinant. Inserting (86) and dropping the Riemann tensor terms, which are sub-leading in the λ -expansion, give

现在可以得到式 (84) 中的度规 $\breve{g}^{\alpha\beta}(x)$ 以及对应的度规行列式。代入 (86) 并舍弃在 λ 展开中次要的黎曼张量项后，得到

$$\begin{aligned}\mathcal{L}_\varphi &= \frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\nu}\partial_\mu\hat{\chi}_c\partial_\nu\hat{\chi}_c \\ &= \sqrt{-\hat{g}_s}\left(\frac{1}{2}n_+\partial\hat{\chi}_c n_- D_s\hat{\chi}_c + \frac{1}{2}\partial_{\alpha_\perp}\hat{\chi}_c\partial^{\alpha_\perp}\hat{\chi}_c\right).\end{aligned} \tag{107}$$

The last line follows from (99) and shows that after the multipole expansion, the leading soft interactions are collected into the covariant derivative.

最后一行由式 (99) 得出，表明多极展开后，领头阶软相互作用都被归入协变导数中。

Applying this procedure to the background field Lagrangian, now including nonlinear terms in the collinear graviton field and keeping the Riemann tensor terms in (86), one finds the soft-collinear Lagrangian in the form

将该步骤应用于背景场拉格朗日量 (此时包含共线引力子场的非线性项，并保留式 (86) 中的黎曼张量项)，可以得到如下形式的软共线拉格朗日量:

$$\mathcal{L} = \sqrt{-\hat{g}_s}\left(\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots\right), \tag{108}$$

where

其中

$$\mathcal{L}^{(0)} = \frac{1}{2} n_+ \partial \hat{\chi}_c n_- D_s \hat{\chi}_c + \frac{1}{2} \partial_{\alpha_\perp} \hat{\chi}_c \partial^{\alpha_\perp} \hat{\chi}_c, \tag{109}$$

$$\mathcal{L}^{(1)} = -\frac{\kappa}{2} \hat{\mathfrak{h}}^{\mu\nu} \partial_\mu \hat{\chi}_c \partial_\nu \hat{\chi}_c + \frac{\kappa}{4} \hat{\mathfrak{h}}^{\beta_\perp}{}_{\beta_\perp} \left(n_+ \partial \hat{\chi}_c n_- D_s \hat{\chi}_c + \partial_{\alpha_\perp} \hat{\chi}_c \partial^{\alpha_\perp} \hat{\chi}_c \right), \tag{110}$$

$$\begin{aligned} \mathcal{L}^{(2)} = & -\frac{1}{8} x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-} (n_+ \partial \hat{\chi}_c)^2 + \frac{\kappa}{2} \hat{\mathfrak{h}}^{\mu\alpha} \hat{\mathfrak{h}}_\alpha{}^\nu \partial_\mu \hat{\chi}_c \partial_\nu \hat{\chi}_c - \frac{\kappa^2}{4} \hat{\mathfrak{h}}^{\alpha_\perp}{}_{\alpha_\perp} \hat{\mathfrak{h}}^{\mu\nu} \partial_\mu \hat{\chi}_c \partial_\nu \hat{\chi}_c \\ & + \frac{\kappa^2}{16} \left(\left(\hat{\mathfrak{h}}^{\alpha_\perp}{}_{\alpha_\perp} \right)^2 - 2 \hat{\mathfrak{h}}^{\alpha\beta} \hat{\mathfrak{h}}_{\alpha\beta} \right) \left(n_+ \partial \hat{\chi}_c n_- D_s \hat{\chi}_c + \partial_{\mu_\perp} \hat{\chi}_c \partial^{\mu_\perp} \hat{\chi}_c \right). \end{aligned} \tag{111}$$

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[10] In (105) – (108), $g_{s\mu\nu}$ and its determinant are evaluated at x, not x_-.

[10] 在式 (105) 中，– (108), $g_{s\mu\nu}$ 及其行列式在 x 处取值，而非 x_-。

The superscript on $\mathcal{L}$ denotes the lowest order in the λ-expansion at which the respective terms contribute. By expanding out the implicit collinear Wilson lines and $n_- D_s$, each term generates an infinite tower of higher-order terms in λ. The derivation of the above Lagrangian is presented in detail in [11], where one can also find the explicit λ-expansion of the Lagrangian up to $\mathcal{O}(\lambda^2)$ in terms of the elementary fields $\hat{\varphi}_c, \hat{h}_{\mu\nu}$, and $s_{\mu\nu}$. In general, there is also the soft matter field φ_s; however, in the absence of scalar self-interactions, it does not contribute to the soft-collinear Lagrangian at $\mathcal{O}(\lambda^2)$.

$\mathcal{L}$ 上标表示对应项贡献的 λ 展开最低阶。通过展开隐含的共线威尔逊线和 $n_- D_s$，每一项都会生成 λ 的无穷高阶项塔。上述拉格朗日量的推导详见文献 [11]，在该文献中还可以找到拉格朗日量以基本场 $\hat{\varphi}_c, \hat{h}_{\mu\nu}$ 和 $s_{\mu\nu}$ 表示、直到 $\mathcal{O}(\lambda^2)$ 阶的显式 λ 展开。一般情况下还存在软物质场 φ_s；但在没有标量自相互作用时，它在 $\mathcal{O}(\lambda^2)$ 阶对软共线拉格朗日量没有贡献。

The soft graviton field $s_{\mu\nu}$ appears above only inside the soft covariant derivative $n_- D_s$, inside the Riemann tensor and the gauge-invariant building blocks. This highlights a formal similarity of the gravitational soft-collinear interactions to the respective gauge theory result [6], featuring a leading interaction via a covariant derivative and sub-leading interactions starting from the quadrupole term (in contrast to the dipole term in gauge theory). The main difference in gravity is that the covariant derivative contains not one but two gauge fields.

软引力子场 $s_{\mu\nu}$ 在上文中仅出现在软协变导数 $n_- D_s$ 内部、黎曼张量以及规范不变构造块中。这体现出引力软共线相互作用与规范理论对应结果形式上的相似性 [6]，即领头阶相互作用通过协变导数实现，次领头阶相互作用从四极项开始 (与规范理论中的偶极项不同)。引力的主要区别在于，协变导数中包含两个而非一个规范场。

Moreover, the Lagrangian above is expressed in terms of the gauge-invariant building blocks $\hat{\chi}_c$ and $\mathfrak{h}_{\mu\nu}$. Similar to the purely collinear Lagrangian (59) in section "Collinear Gravity," the theory is invariant under the redefined hatted-collinear transformations, and one can decide to employ either the gauge-invariant or non-invariant fields $\hat{\varphi}_c$ and $\hat{h}_{\mu\nu}$. With the latter choice, one finds that the collinear Wilson line W_c^{-1} drops out of all terms that do not contain the soft Riemann tensor. Above, this applies to all but the first term in (111), since this term is not minimally coupled to the homogeneous background field [11] and would consequently contain explicit factors of W_c^{-1} .

此外，上述拉格朗日量是用规范不变构造块 $\hat{\chi}_c$ 和 $\mathfrak{h}_{\mu\nu}$ 表示的。与“共线引力”一节中的纯共线拉格朗日量 (59) 类似，该理论在重新定义的带帽共线变换下不变，我们可以选择使用规范不变场或非规范不变场 $\hat{\varphi}_c$ 和 $\hat{h}_{\mu\nu}$ 。选择后者时会发现，共线威尔逊线 W_c^{-1} 会从所有不含软黎曼张量的项中消去。上文里，这适用于 (111) 中除第一项外的所有项，因为该项并非最小耦合于均匀背景场 [11]，因此会显式包含 W_c^{-1} 因子。

In the purely collinear sector (setting all soft fields to zero), gravity is distinct from gauge theory. There is no collinear-covariant derivative D_c^μ in gravity, and correspondingly, leading-power collinear interactions are absent. In combination with the operator basis discussed below, where the first graviton building block starts at $\mathcal{O}(\lambda)$, this implies that one cannot generate collinear emissions at leading power. In consequence, there are no jets in gravity, and each additional collinear emission costs a power of λ . This immediately shows that gravity does not exhibit any collinear divergences, and indeed nothing does go wrong in gravity [41].

在纯共线区域 (令所有软场为零)，引力与规范理论不同。引力中不存在共协变导数 D_c^μ ，相应地也没有领头幂次的共线相互作用。结合下文讨论的算符基 (其中第一个引力子构造块从 $\mathcal{O}(\lambda)$ 开始)，这意味着我们无法产生领头幂次的共线辐射。因此，引力中不存在喷注，每增加一次共线辐射都需要付出一个 λ 幂次的代价。这直接说明引力不存在任何共线发散，引力中确实不会出现问题 [41]。

Moreover, in the soft-collinear sector, the covariant derivative can be eliminated iteratively from the subleading-power Lagrangians by making use of the field equation, such that it appears only in $\mathcal{L}^{(0)}$. Therefore, gravity features an extended eikonal interaction compared to gauge theory, related to the two independent gauge fields in n_-D_s . This fundamental result follows quite naturally from the SCET gravity construction.

此外，在软共线区域，协变导数可以利用场方程从次领头幂次拉格朗日量中逐次消去，最终仅出现在 $\mathcal{L}^{(0)}$ 中。因此，相比规范理论，引力具有推广的程函相互作用，这与 n_-D_s 中两个独立规范场有关。这一基础结果自然源自 SCET 引力构造。

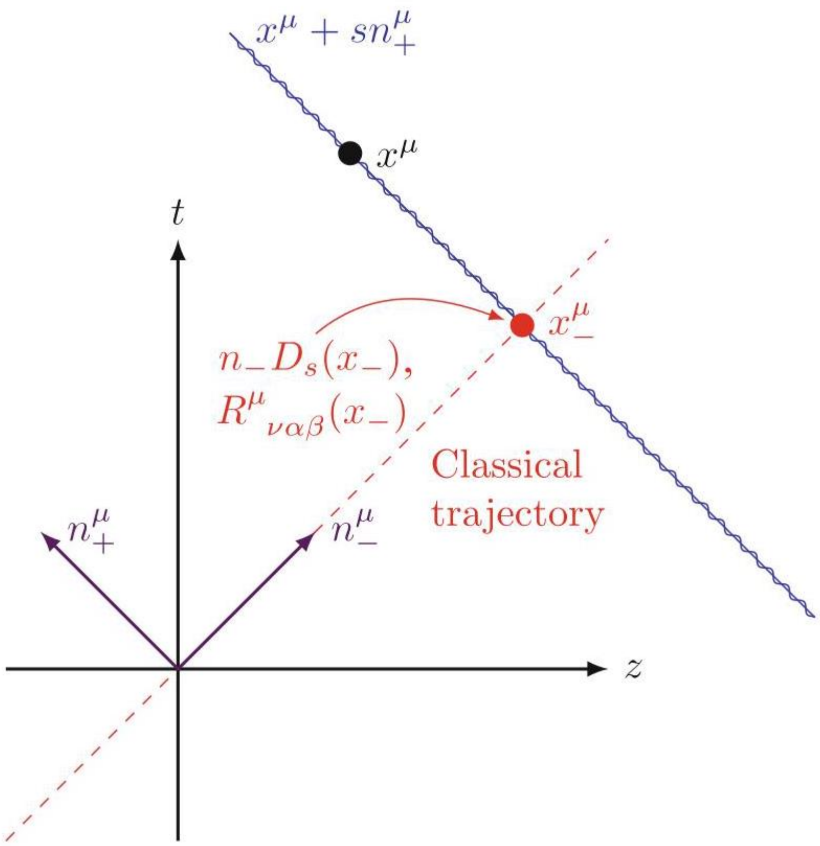

Fig. 4 See text for explanation

图 4 说明见正文

Figure 4 displays the space-time structure of SCET gravity interactions, showing for simplicity only a single collinear direction and suppressing the transverse directions. The classical trajectory of a massless, energetic (collinear) particle passing through $\mathbf{x} = 0$ at $t = 0$ is the line $x_-^\mu = (n_+ x)\frac{n_-^\mu}{2}$ (dashed red). Collinear field operators describing these particles are non-local in the opposite light-like direction n_+^μ (wavy blue) as discussed around (24) and section "Sources and Hard Matching," but interact with soft fields only at x_-^μ via the extended eikonal interaction and the manifestly covariant Riemann tensor terms as a consequence of the light-front multipole expansion.

图 4 展示了 SCET 引力相互作用的时空结构，为简化仅画出单个共线方向，省略了横方向。一个无质量高能 (共线) 粒子穿过 $\mathbf{x} = 0$ 中 $t = 0$ 的经典轨迹是直线 $x_-^\mu = (n_+ x)\frac{n_-^\mu}{2}$ (红色虚线)。如 (24) 式附近以及“源与硬匹配”一节的讨论，描述这类粒子的共场算符在相反类光方向 n_+^μ (蓝色波浪线) 上非定域，但由于光前多极展开，它们仅在 x_-^μ 处通过推广的程函相互作用和显式协变黎曼张量项与软场相互作用。

Gauge-Invariant Building Blocks

规范不变构造块

In the construction of the Lagrangian, we identified a set of gauge-invariant operators that can be used as ingredients in the hard matching. The only collinear gauge-invariant building blocks in the operator basis are (cf. (103), (104))

在拉格朗日量的构造中，我们得到了一组可作为硬匹配中组分的规范不变算符。算符基中仅有的共线规范不变构造块如下 (参见式 (103)、(104))

$$\hat{\chi}_c = \left[W_c^{-1}\hat{\varphi}_c\right]$$

$$\kappa\hat{\mathfrak{h}}_{\mu\upsilon} = W_\mu^\alpha W_\upsilon^\beta \left[W_c^{-1}\kappa\hat{h}_{\alpha\beta}\right] + \left(W_\mu^\alpha W_\upsilon^\beta \left[W_c^{-1}\hat{g}_{s\alpha\beta}(x)\right] - \hat{g}_{s\mu\upsilon}(x)\right),$$

(112)where $\mu\upsilon \in \{\perp\perp, \perp-, --\}$, thanks to the condition $\hat{\mathfrak{h}}_{\mu+} = 0$ defining the collinear Wilson line. The sub-leading minus components $\hat{\mathfrak{h}}_{\perp-}$ and $\hat{\mathfrak{h}}_{--}$ can be eliminated using the collinear equation of motion, which, at leading order, express these components in terms of derivatives of $\mathfrak{h}_{\perp\perp}$, similar to (62) in the purely collinear theory. The building blocks $\hat{\chi}_c$ and $\hat{\mathfrak{h}}_{\perp\perp}$ both scale as $\mathcal{O}(\lambda)$. Additional suppression in λ is obtained by acting with $\partial_\perp^\mu$ on collinear building blocks. Each additional derivative increases the λ -power by one unit. The ordinary derivative is the appropriate one, because one operates on gauge-invariant field products.

其中 $\mu\upsilon \in \{\perp\perp, \perp-, --\}$，这得益于定义共线威尔逊线的条件 $\hat{\mathfrak{h}}_{\mu+} = 0$ 。次 leading 的减分量 $\hat{\mathfrak{h}}_{\perp-}$ 和 $\hat{\mathfrak{h}}_{--}$ 可以通过共线运动方程消去，在 leading 阶，运动方程将这些分量表示为 $\mathfrak{h}_{\perp\perp}$ 的导数，与纯共线理论中式 (62) 类似。构造块 $\hat{\chi}_c$ 和 $\hat{\mathfrak{h}}_{\perp\perp}$ 的标度均为 $\mathcal{O}(\lambda)$ 。通过将 $\partial_\perp^\mu$ 作用在共线构造块上，可以得到 λ 中额外的压低。每多一个导数，λ 的幂次就增加一个单位。普通导数是适用的，因为我们作用的对象是规范不变场乘积。

While the building blocks are inherently inhomogeneous in λ , one can use the power counting of the leading term to characterize these operators. The effect of the infinite tower of terms sub-leading in λ is simply to render the leading term gauge-invariant. If one fixes collinear light cone gauge, the entire sub-leading tower disappears, and one obtains homogeneously scaling operators.

尽管这些构造块本身在 λ 下是非齐次的，但我们可以利用领头项的幂计数来刻画这些算符。无穷多 λ 次 lead 项的作用只是让领头项成为规范不变的。如果固定共线光锥规范，所有次 lead 项都会消失，我们得到齐次标度的算符。

For the soft sector, one employs gauge-covariant building blocks, like the soft scalar φ_s , the derivative n_-D_s , or the Riemann tensor $R^\mu{}_{\nu\alpha\beta}$. However, the soft covariant derivative can be eliminated from the building blocks by the collinear field equations [11]. For the scalar field, at leading power

对于软区域，我们采用规范协变构造块，例如软标量 φ_s 、导数 n_-D_s ，或是黎曼张量 $R^\mu{}_{\nu\alpha\beta}$ 。不过，可以通过共线场方程从构造块中消去软协变导数 [11]。对于标量场，在领头幂次下

$$n_-D_s\hat{\chi}_c = -\frac{\partial_\perp^2}{n_+\partial}\hat{\chi}_c, \tag{113}$$

and a similar equation can be derived from the Einstein-Hilbert Lagrangian. This shows that $n_-D_s\hat{\chi}_c$ and $n_-D_s\mathfrak{h}_{\perp\perp}$ are redundant [11] and can be traded for (nonlocal) combinations of the other collinear and soft building blocks, which will contribute to the equation of motion at higher powers. This implies that the soft covariant derivative is not a relevant building block for the N -jet operators (sources). When n_D_s is eliminated systematically, the remaining soft graviton building block is the Riemann tensor and its derivatives. Therefore, soft gauge covariance implies that soft graviton building blocks are suppressed by at least λ^6 · [11]

并且我们可以从爱因斯坦-希尔伯特拉格朗日量推导出类似的方程。这说明 $n\_D_s\hat{\chi}_c$ 和 $n\_D_s\mathfrak{h}_{\perp\perp}$ 是冗余的 [11]，可以替换为其他共线和软构造块的 (非定域) 组合，这些组合会在更高幂次下对运动方程产生贡献。这说明软协变导数不是 N 喷注算符 (源) 的相关构造块。当系统地消去 n_D_s 后，剩余的软引力子构造块是黎曼张量及其导数。因此，软规范不变性要求软引力子构造块至少被 λ^6 · [11] 压低

Sources and Hard Matching

源项与硬匹配

The aim of the previous sections was to construct the Lagrangian of an effective theory that reproduces the gravitational scattering amplitudes in the soft and collinear limits, that is, when some of the external momenta k_i form small invariants [12] $k_i \cdot k_j \ll Q^2$, where Q is the scale of the hard scattering. The soft-collinear effective Lagrangian captures only part of the full scattering amplitude: it describes purely collinear interactions within a single collinear direction and the soft-collinear interactions. To recover the full scattering amplitude, one also needs to include the effect of the highly virtual propagators and loops, the ”hard” region. Hard lines and loops always connect different directions and are the source of large-angle scattering of energetic particles in the first place. In the pictorial representation of Fig. 2, the entire hard process is shrunk to the central point.

前面几节的目标是构造有效理论的拉格朗日量，使其能够重现软极限和共线极限下的引力散射振幅；也就是说，当部分外动量 k_i 形成小不变量 [12] $k_i \cdot k_j \ll Q^2$ ，其中 Q 为硬散射的能标时，该拉格朗日量依然成立。软共线有效拉格朗日量仅能捕捉完整散射振幅的一部分: 它仅描述单个共线方向内的纯共线相互作用，以及软共线相互作用。要得到完整的散射振幅，还需要纳入高虚 propagator 和圈，即“硬”区的效应。硬线和硬圈始终连接不同方向，它们本身就是高能粒子大角散射的来源。在图 2 的示意图中，整个硬过程被收缩到中心点。

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[11] There is an implicit dependence on the soft metric field in the definition of the collinear building blocks (112) through W_c^{-1} and $\hat{g}_{s\mu\nu}$. This dependence is already constrained by collinear gauge invariance and does not give rise to soft building blocks. The soft graviton vertices generated by this dependence always contain collinear gravitons as well.

[11] 在共线构造块 (112) 的定义中，通过 W_c^{-1} 和 $\hat{g}_{s\mu\nu}$ 隐含依赖于软度规场。该依赖已经被共线规范不变性约束，不会产生软构造块。由这种依赖产生的软引力子顶点也始终包含共线引力子。

[12] Lorentz-invariant products of different momenta

[12] 不同动量的洛伦兹不变乘积

In technical terms, in the scattering of multiple collinear sectors, the intermediate highly virtual states are hard modes due to the presence of the different large momentum components $n_{i+}p_i, n_{j+}p_j, \dots$, from which one can form hard invariants

从技术层面讲，多个共线区的散射过程中，由于存在不同的大动量分量 $n_{i+}p_i, n_{j+}p_j, \dots$ ，中间高虚态是硬模，由此可以构成硬不变量

$$(n_i + p_i)(n_{j+}p_j)(n_{i-}n_{j-}) \sim \lambda^0. \tag{114}$$

The hard region is integrated out in the EFT by construction and is then encoded in the hard matching coefficients of the operators containing fields of multiple collinear directions. This gives rise to the so-called N -jet operators or ”currents” or ”sources”. [13] The full scattering amplitude is then the sum of matrix elements of N -jet operators evaluated with the soft-collinear effective Lagrangian. The construction of the effective Lagrangian is a prerequisite to the construction of N - jet operators, because it defines what is the field basis from which the hard sources are constructed and the symmetries they must respect. After having established this in the previous sections, the admissible sources can now be easily specified. Their matching coefficients depend on the particular scattering process and must of course be calculated in a given context.

根据有效场论的构造，硬区已经被积出，其效应被编码在包含多个共线方向场的算符的硬匹配系数中。这就产生了所谓的 N 喷注算符，也叫“流”或“源”。[13] 此时完整散射振幅就是用软共线有效拉格朗日量计算得到的 N 喷注算符矩阵元之和。有效拉格朗日量的构造是构造 N 喷注算符的前提，因为它定义了构造硬源所用的场基底，以及硬源必须满足的对称性。我们已经在前几节确立了这些基础，现在可以很容易地给出容许源的具体形式。它们的匹配系数依赖于具体的散射过程，当然必须在特定背景下计算。

The N -jet operators are constructed from the collinear and soft building blocks. It corresponds to a purely hard scattering that produces the constituent building block particles of the operator. For example, the tree-level scattering of N widely separated energetic scalar particles is encoded in a current operator consisting of N copies of the collinear gauge-invariant scalar $\hat{\chi}_{c_i}$, one for each direction, combined with a hard matching coefficient. One can then think of the hard matching coefficient as the nonradiative amplitude of the hard process, while the building blocks correspond to the external legs of the amplitude.

N 喷注算符由共线构造块和软构造块构造而来，它对应产生算符组成构造块粒子的纯硬散射。例如，N 个大间距高能标量粒子的树级散射被编码在一个流算符中: 该流算符包含 N 份共线规范不变标量 $\hat{\chi}_{c_i}$ (每个方向一份)，再结合一个硬匹配系数。我们可以把硬匹配系数看作硬过程的非辐射振幅，而构造块对应振幅的外腿。

This notion is made more precise in the following. More details for the gauge theory situation can be found in [7] and for the gravitational case in [11]. The generic N -jet operator in gauge theory is a light-ray operator [7]

下文会将这一概念表述得更精确。规范理论情形的更多细节可以参考文献 [7]，引力情形的更多细节可以参考文献 [11]。规范理论中一般的 N 喷注算符是光射线算符 [7]

$$\mathcal{J}(0) = \int [dt]_N \widetilde{C}(t_{i_1}, t_{i_2}, \dots) J_s(0) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots), \tag{115}$$

$[dt]_N = \prod_{i_k} dt_{i_k}$, and " 0 " refers to the origin $x = 0$, where the hard scattering is supposed to take place. Here, $\widetilde{C}(t_i)$ is the hard matching coefficient, and J_i are composite operators of i -collinear fields and J_s of soft fields. The variables t_{i_k} correspond to the non-locality of collinear operators along the light cone, for example, $J_i(t_{i_1}, t_{i_2}, \dots)$ might refer to the product $\hat{\chi}_{c_i}(t_{i_1} n_{i+}) \hat{\chi}_{c_i}(t_{i_2} n_{i+}) \dots$ of fields in the same collinear direction. In momentum space, $t_{i_k} \to n_{i+} \cdot p_{i_k}$, which can be expressed in terms of the fraction of total momentum in direction i carried by the factor k in J_i , which provides the relation to the intuitive notion of the non-radiative amplitude.

$[dt]_N = \prod_{i_k} dt_{i_k}$ ，其中“0”指硬散射假定发生的原点 $x = 0$ 。此处，$\widetilde{C}(t_i)$ 是硬匹配系数，J_i 是 i 共线场与软场的 J_s 构成的复合算符。变量 t_{i_k} 对应共线算符沿光锥的非定域性，例如 $J_i(t_{i_1}, t_{i_2}, \dots)$ 可指代同一共线方向上场乘积 $\hat{\chi}_{c_i}(t_{i_1} n_{i+}) \hat{\chi}_{c_i}(t_{i_2} n_{i+}) \dots$ 。在动量空间中，$t_{i_k} \to n_{i+} \cdot p_{i_k}$ 可以用 J_i 中因子 k 携带的 i 方向总动量占比表示，这建立了其与非辐射振幅直观概念的联系。

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[13] The reason why the N -jet operators do not appear in the Lagrangian as interactions connecting different collinear sectors is that they can appear in a hard process only once, i.e., the hard subgraph must be connected, provided that there are no collinear particles in the same direction in the initial and final state of the scattering. In this case, collinear modes in different directions can never "meet again" in a hard process as a consequence of the Coleman-Norton theorem [23].

[13] N 喷注算符不出现在拉格朗日量中作为连接不同共线区的相互作用，原因是它们在硬过程中只能出现一次: 即只要散射初态和末态不存在同方向的共线粒子，硬子图就必须是连通的。在此情况下，根据 Coleman-Norton 定理 [23]，不同方向的共线模式永远不会在硬过程中“再次相遇”。

Like the full theory scattering amplitude, these N -jet operators must be gauge-invariant. In the effective theory, this means invariance under both the soft and the collinear gauge symmetries. To ensure collinear gauge invariance, one uses the gauge-invariant building blocks in the collinear current J_i , explicitly given in (112) in the previous section. Soft fields are automatically collinear gauge-invariant. In addition, the collinear building blocks are soft gauge-covariant, as they are constructed from the hatted fields. For J_s , one should employ soft covariant building blocks like φ_s or the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$, as discussed in section "Gauge-Invariant Building Blocks." The N -jet operator as defined in (115) is now collinear gauge-invariant and soft gauge-covariant, transforming under global soft Poincaré transformations $U_s(0)$. The N -jet operator as defined in (115) is located at $x = 0$, however, and hence not translation-invariant. To render it manifestly gauge-invariant under both soft and collinear gauge transformations, one defines the translation- and therefore gauge-invariant current

和完整理论的散射振幅一样，这些 N 喷注算符必须满足规范不变性。在有效理论中，这意味着它同时在软规范对称性和共线规范对称性下不变。为保证共线规范不变性，人们使用共流 J_i 中的规范不变构造块，其上一节的 (112) 式已明确给出。软场自动满足共线规范不变性。此外，共线构造块是软协变规范的，因为它们由带帽场构造得到。对于 J_s，应当采用如 φ_s 或黎曼张量 $R^{\mu}{}_{\nu\alpha\beta}$ 这样的软协变构造块，正如"规范不变构造块"一节所讨论的。(115) 式定义的 N 喷注算符现在满足共线规范不变性和软协变规范，在整体软庞加莱变换 $U_s(0)$ 下协变。不过 (115) 式定义的 N 喷注算符定域在 $x=0$，因此不满足平移不变性。为了让它在软规范变换和共线规范变换下都明显规范不变，人们定义了同时满足平移不变、因而也规范不变的流

$$\mathcal{J} = \int d^4x T_x \mathcal{J}(0)\, T_x^{-1}, \tag{116}$$

where $T_x = e^{ix\hat{p}}$ is the translation operator. In practical computations, the integral and translation operators reduce to the momentum conserving δ -function once the amplitude is evaluated. Therefore, one can choose the simpler N -jet operator (115) located at $x = 0$ and impose momentum and angular momentum conservation by hand, similar to imposing color neutrality in QCD.

其中 $T_x = e^{ix\hat{p}}$ 是平移算符。在实际计算中，当计算振幅时，积分和平移算符会退化为动量守恒的 δ 函数。因此，可以选用定域在 $x = 0$ 的更简单的 N 喷注算符 (115)，再手动施加动量和角动量守恒，这类似于 QCD 中手动施加色中性条件。

As said above, the hard coefficient is obtained from a process-specific matching computation. To perform this matching, one follows the standard method of evaluating a suitable on-shell Green function in the EFT and in the full theory and demanding that they are equal up to a desired power in the λ -expansion. One typically chooses the simplest possible external states that give non-vanishing matrix elements in both theories. Once the matching coefficients are determined, one can use the EFT operators to evaluate arbitrarily complex low-energy matrix elements. Note that these sources are the only place in the effective theory where renormalization takes place. The Lagrangian is not renormalized [5], in the sense that no further renormalization is required, when the SCET Lagrangian is expressed in terms of the renormalized full theory couplings. [14] The matching coefficients, however, receive loop corrections.

如上所述，硬系数由特定过程的匹配计算得到。要完成该匹配，需遵循标准方法: 分别在有效场论 (EFT) 和完整理论中计算合适的在壳格林函数，并要求二者在 λ 展开的期望幂次精度内相等。通常我们会选取能让两个理论都得到非零矩阵元的最简单外态。一旦确定了匹配系数，就可以利用 EFT 算符计算任意复杂度的低能矩阵元。请注意，这些源是有效理论中唯一发生重整化的地方。当 SCET 拉格朗日量以重整化后的完整理论耦合表示时，拉格朗日量本身不需要额外重整化 [5]。[14] 但匹配系数会得到圈修正。

Soft Theorem

软定理

If the radiative amplitude for the emission of (usually) a single soft particle from a hard scattering process can be expressed in terms of the non-radiative one without detailed knowledge of the latter, one refers to the corresponding result as a "soft theorem." The best-known and earliest example of a soft theorem is the abelian version of (3), which expresses the soft photon emission amplitude in QED in terms of a universal ”eikonal factor.” In QED, the absence of photon self-interactions leads to the exponentiation of multiple soft photon emissions. Of particular relevance to this chapter is the further fact, known as the Low-Burnett-Kroll theorem [20,32], that the soft theorem extends to next-to-leading order in the soft expansion.

> 如果从硬散射过程辐射出 (通常为) 单个软粒子的振幅，可以在不需要非辐射振幅详细信息的情况下用非辐射振幅表示，我们就将该结果称为“软定理”。最著名、最早的软定理例子是 (3) 的阿贝尔版本，它用通用的“程函因子”表示量子电动力学中的软光子发射振幅。在量子电动力学中，光子不存在自相互作用，因此多重软光子发射可以指数化。对于本章尤为重要的一个结论是洛-伯内特-克罗尔定理 [20,32]，即软定理可以拓展到软展开的次领头阶。

Almost 50 years after Weinberg’s first discussion of the gravitational soft theorem (2), Cachazo and Strominger [21] made the remarkable observation that its universality extends by two orders in the soft expansion. If $\mathcal{A}(\{p_i\})$ denotes the amplitude for a N -particle hard scattering process with momenta $p_1, \ldots, p_N$, the single graviton emission amplitude at tree level reads

> 在温伯格首次讨论引力软定理 (2) 近 50 年后，卡查佐和斯特罗明格 [21] 得到了一项惊人发现: 软定理的普遍性可以在软展开中多拓展两阶。若 $\mathcal{A}(\{p_i\})$ 表示动量为 $p_1, \ldots, p_N$ 的 N 粒子硬散射过程的振幅，则树图阶单个引力子发射振幅可写为

$$\mathcal{A}_{\text{rad}}(\{p_i\};k) = \frac{\kappa}{2}\sum_{i=1}^{N}\bar{u}(p_i)\Bigg(\frac{\varepsilon_{\mu\upsilon}(k)\,p_i^{\mu}p_i^{\upsilon}}{p_i\cdot k} + \frac{\varepsilon_{\mu\upsilon}(k)\,p_i^{\mu}k_{\rho}J_i^{\upsilon\rho}}{p_i\cdot k} + \frac{1}{2}\frac{\varepsilon_{\mu\upsilon}(k)\,k_{\rho}k_{\sigma}J_i^{\rho\mu}J_i^{\sigma\upsilon}}{p_i\cdot k} + \mathcal{O}(k^2)\Bigg)\mathcal{A}(\{p_i\}). \tag{117}$$

Here,

> 在此处，

$$J_i^{\mu\upsilon} = L_i^{\mu\upsilon} + \sum\nolimits_i^{\mu\upsilon} = p_i^{\mu}\frac{\partial}{\partial p_{i\upsilon}} - p_i^{\upsilon}\frac{\partial}{\partial p_{i\mu}} + \sum\nolimits_i^{\mu\upsilon} \tag{118}$$

refers to generators of the Lorentz group, in this context usually referred to as the ”angular momentum operator,” with $L_i^{\mu\upsilon}$ the orbital angular momentum operator of particle i from which the graviton is emitted and $\sum_i^{\mu\upsilon}$ the spin operator in the representation of emitter (”matter”) particle i.[15] $\bar{u}(p_i)$ denotes the polarization functions of the matter particles and $\varepsilon_{\mu\upsilon}(k)$ the graviton polarization tensor. In the following subsections, we shall focus on scalar matter, in which case $J_i^{\mu\upsilon} = L_i^{\mu\upsilon}$ and the $\bar{u}(p_i)$ are trivial. It is worth noting that one recovers the Low-Burnett-Kroll theorem for the emission of a photon, and its non-abelian generalization [22], from (117) by substituting $p_i^{\mu} \to t_i^a, \varepsilon_{\mu\upsilon}(k) \to \varepsilon_{\upsilon}(k)$, and $\kappa/2 \to -g_s$ and dropping the next-to-next-to-soft term in the second line of (117), which is unique to gravitons. The above substitutions are suggestive of color-kinematics duality, but we shall see below that regarding the next-to-soft terms in gauge theory and gravity as analogues of each other does not correspond to the proper interpretation of their physics origin.

指代洛伦兹群的生成元，在此语境中通常被称为“角动量算符”，其中 $L_i^{\mu\upsilon}$ 是发射引力子的粒子 i 的轨道角动量算符，$\sum_i^{\mu\upsilon}$ 是发射源（“物质”）粒子 i.[15]$\bar{u}(p_i)$ 表示中自旋算符，i.[15]$\bar{u}(p_i)$ 指代物质粒子的极化函数，$\varepsilon_{\mu\upsilon}(k)$ 指代引力子极化张量。在接下来的小节中，我们将聚焦标量物质，此时 $J_i^{\mu\upsilon} = L_i^{\mu\upsilon}$ 且 $\bar{u}(p_i)$ 是平庸的。值得注意的是，对光子发射而言，我们可以通过替换 $p_i^\mu \to t_i^a, \varepsilon_{\mu\upsilon}(k) \to \varepsilon_\upsilon(k)$ 和 $\kappa/2 \to -g_s$，并去掉 (117) 第二行中引力子特有的次次软项，从 (117) 得到洛-伯内特-克罗尔定理及其非阿贝尔推广 [22]。上述替换暗示了色运动学对偶，但我们下文会说明，将规范理论和引力中的次软项看作相互对应，并不能正确解释它们的物理起源。

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[14] In gravity, one has to keep in mind that the renormalized full theory is constructed up to a certain loop order by introducing additional higher derivative operators, as explained in section ”Perturbative Gravity.”

[14] 在引力中，我们必须牢记: 如“微扰引力”章节所述，重整化后的完整理论是通过引入额外的高阶导数算子，构造到特定圈阶的。

[15] All up to a factor of i , omitted for the convenience of writing (117) without factors of i

[15] 所有项都相差一个 i 因子，为方便书写 (117，我们省略了该因子，不保留 i 的相关系数

The discovery of the sub-sub-leading gravitational soft theorem (117) was inspired [39, 40] by a relation between the leading soft theorem and the Bondi, van der Burg, Metzner, and Sachs symmetries [18,36], which consist of a subgroup of diffeomorphisms operating on the asymptotically flat regions of space-times, the significance of which has not yet become completely clear. The actual derivation of (117) in [21] used the spinor-helicity formalism, which is particularly elegant for matter particles with spin, as the angular momentum operator in spinor-helicity variables combines the orbital and spin parts in a simple way. Subsequently, the next-to-next-to-soft theorem was derived in various other ways [16, 19,44].

次次领头阶引力软定理 (117) 的发现灵感 [39, 40] 来源于领头软定理与邦迪-范德布堡-梅茨纳-萨克斯对称性之间的关系 [18,36]，该对称性是作用在时空渐近平坦区域的微分同胚子群，其重要性至今尚未被完全阐明。[21] 中对 (117) 的实际推导使用了旋量螺旋度形式，该方法对带自旋的物质粒子格外简洁，因为旋量螺旋度变量中的角动量算符可以将轨道部分和自旋部分简单结合起来。之后，人们又通过多种不同方法推导出了次次软定理 [16, 19,44]。

In the spirit of the article, we ask what the effective field theory can say about the soft theorem. Obviously, the EFT must reproduce it as the special case of single soft emission, but can it provide additional insight? In the previous sections, the important role of the emergent gauge symmetries in the construction of soft-collinear gravity has become apparent. The multipole expansion of the Lagrangian naturally leads to structures that already resemble the angular momentum operator, which shows up explicitly in the coupling to one of the gauge fields. In the following, we therefore focus on the fundamental questions: Why are there three and exactly three universal terms in the soft theorem? What are the origin and interpretation of the angular momentum factors? Is the soft theorem corrected by loop effects?

秉承本文的思路，我们探讨有效场论能对软定理给出什么结论。显然，有效场论 (EFT) 必然能将软定理作为单次软发射的特殊情况重现，但它能否带来额外的洞见？在前文中，涌现规范对称性在软共线引力构造中的重要作用已经十分明晰。拉格朗日量的多极展开自然得到了形式上与角动量算子相似的结构，该结构在与某一规范场的耦合中会明确显现。因此在下文中，我们聚焦以下核心问题: 软定理中为何存在三个，且恰好是三个普适项？角动量因子的起源与物理意义是什么？软定理是否会被圈效应修正？

Before approaching these questions from soft-collinear gravity, the essence of the derivation of (117) from the explicit expansion of the scattering amplitude will be briefly reviewed, following closely [16].

在从软共线引力出发探讨这些问题之前，我们将紧密遵循文献 [16]，简要回顾从散射振幅显式展开推导 (117) 式的核心内容。

Soft Theorem from Graviton Amplitudes

从引力子振幅导出软定理

Since amplitudes factorize over their poles, the $N+1$ -point amplitude with emission of a single soft graviton at tree level from a hard N -particle scattering process with amplitude $\mathcal{A}(\{p_i\})$ can be written as

由于振幅可通过极点因子化，树图水平下，硬过程 N 粒子散射振幅为 $\mathcal{A}(\{p_i\})$，从中辐射出单个软引力子的 $N+1$ 点振幅可以写为

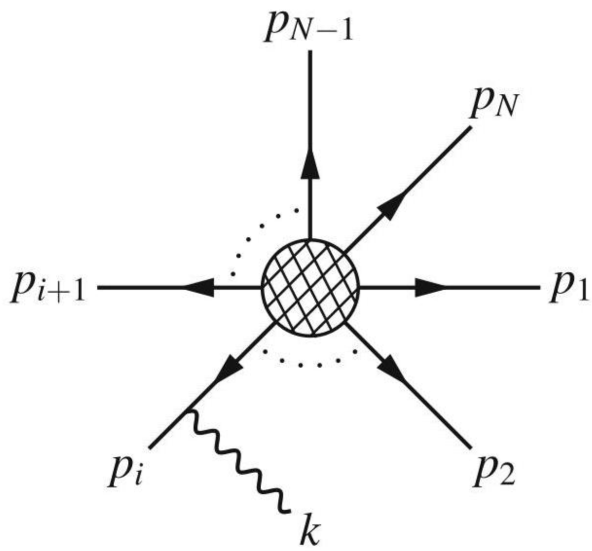
$$\mathcal{A}^{\mu\upsilon}_{\text{rad}}(\{p_i\};k) = \sum_{i=1}^{N} \frac{p_i^\mu p_i^\upsilon}{p_i \cdot k}\mathcal{A}(p_1,\dots,p_i+k,\dots,p_N) + \mathcal{B}^{\mu\upsilon}(\{p_i\};k)\,. \tag{119}$$

The polarization tensor $\varepsilon_{\mu\nu}(k)$ of the graviton has been stripped off, as well as the overall gravitational coupling factor $\kappa/2$. The first term on the right-hand side contains the non-radiative amplitude with one momentum shifted by the graviton momentum k and fully accounts for the singular $\mathcal{O}(1/k)$ term of the amplitude, which arises from the emission off an external leg, as shown in Fig. 5. The second term starts at the next-to-soft order $\mathcal{O}(k^0)$. For simplicity of presentation, it is assumed that the N hard particles have spin 0 .

我们已经分离出引力子的极化张量 $\varepsilon_{\mu\nu}(k)$，以及整体引力耦合因子 $\kappa/2$。右手边第一项包含非辐射振幅，其中一个动量被引力子动量 k 移动，完整解释了来自外腿辐射的振幅奇异 $\mathcal{O}(1/k)$ 项，如图 5 所示。第二项起于次软阶 $\mathcal{O}(k^0)$。为简化表述，假设 N 个硬粒子自旋为 0。

Fig. 5 Emission from the external leg gives the first term in the decomposition of the amplitude in (119)

图 5 外腿辐射给出式 (119) 中振幅分解的第一项



One now makes use of the invariance of on-shell amplitudes under the gauge transformations of weak-field gravity, which implies that $\varepsilon_{\mu\upsilon}(k)\mathcal{A}^{\mu\upsilon}_{\text{rad}}(\{p_i\};k)$ does not change if the graviton polarization tensor is replaced by

> 下面我们利用壳振幅在弱场引力规范变换下的不变性, 这意味着即便将引力子极化张量替换为以下形式，$\varepsilon_{\mu\upsilon}(k)\mathcal{A}^{\mu\upsilon}_{\text{rad}}(\{p_i\};k)$ 也不会发生改变

$$\varepsilon_{\mu\upsilon}(k) \to \varepsilon_{\mu\upsilon}(k) + k_\mu \alpha_\upsilon(\{p_i\};k)\,, \tag{120}$$

where $\alpha_\upsilon(\{p_i\};k)$ is a function of momenta satisfying $k^\upsilon\alpha_\upsilon(\{p_i\};k) = 0$. Since $\alpha_\upsilon(\{p_i\};k)$ is arbitrary, it follows that

> 其中 $\alpha_\upsilon(\{p_i\};k)$ 是满足 $k^\upsilon\alpha_\upsilon(\{p_i\};k) = 0$ 的动量函数。由于 $\alpha_\upsilon(\{p_i\};k)$ 是任意的，因此可得

$$\begin{aligned} 0 &= k_\mu \mathcal{A}^{\mu\nu}_{\text{rad}}(\{p_i\};k) \\ &= \sum_{i=1}^{N} p_i^\upsilon \mathcal{A}(p_1,\ldots,p_i+k,\ldots,p_N) + k_\mu \mathcal{B}^{\mu\upsilon}(\{p_i\};k)\,. \end{aligned} \tag{121}$$

The amplitudes in this equation are analytic in k for tree-level emission. Expanding in k provides relations between (derivatives of) $\mathcal{B}^{\mu\nu}$ and $\mathcal{A}(\{p_i\})$. The vanishing of (121) at $\mathcal{O}\left(k^0\right)$ is guaranteed by momentum conservation, $\sum\limits_{i=1}^{N} p_i^\upsilon = 0$. The next two orders result in

> 该方程中的振幅在树图辐射情况下对 k 解析。对 k 做展开给出 $\mathcal{B}^{\mu\nu}$ 与 $\mathcal{A}(\{p_i\})$ (导数) 之间的关系。动量守恒保证式 (121) 在 $\mathcal{O}\left(k^0\right)$ 处为零，即 $\sum\limits_{i=1}^{N} p_i^\upsilon = 0$ 。接下来两阶给出结果

$$\sum_{i=1}^{N} p_i^\upsilon \frac{\partial}{\partial p_{i\mu}} \mathcal{A}(\{p_i\}) + \mathcal{B}^{\mu\nu}(\{p_i\};0) = 0, \tag{122}$$

$$\sum_{i=1}^{N} p_i^{\nu} \frac{\partial^2}{\partial p_{i\mu} \partial p_{i\rho}} \mathcal{A}(\{p_i\}) + \left[\frac{\partial \mathcal{B}^{\mu\nu}}{\partial k_\rho} + \frac{\partial \mathcal{B}^{\rho\nu}}{\partial k_\mu} \right] (\{p_i\}; 0) = 0. \tag{123}$$

The expansion of (121) does not give the above two equations directly, but multiplied by k_μ and $k_\mu k_\rho$, respectively. Removing these vectors is justified, because any local gauge-invariant term in $\mathcal{B}^{\mu\nu}$ that would be missed in this operation can appear only at order k^2 [16].

对式 (121) 做展开无法直接得到上述两个方程，它们分别乘上了 k_μ 和 $k_\mu k_\rho$ 。消去这些矢量是合理的，因为该操作遗漏的任何 $\mathcal{B}^{\mu\nu}$ 中的局部规范不变项仅会出现在 k^2 阶 [16]。

Gauge invariance thus allows to express the $k \to 0$ limit of $\mathcal{B}^{\mu\nu}$ in terms of the derivative of the non-radiative amplitude, as well as a symmetric first-order derivative of $\mathcal{B}^{\mu\nu}$ in terms of the second derivative. Inserting (122) and (123) into the expansion of (119) in k gives

因此规范不变性允许我们将 $\mathcal{B}^{\mu\nu}$ 的 $k \to 0$ 极限用非辐射振幅的导数，以及对称一阶导数对 $\mathcal{B}^{\mu\nu}$ 用二阶导数表示出来。将 (122) 和 (123) 代入式 (119) 对 k 的展开可得

$$\begin{aligned}
\mathcal{A}_{\text{rad}}^{\mu\nu}(\{p_i\}; k) &= \sum_{i=1}^{N} \frac{p_i^{\nu}}{p_i \cdot k} \left[p_i^{\mu} + k_\rho J_i^{\mu\rho} \right] \mathcal{A}(\{p_i\}) \\
&+ \frac{1}{2} k_\rho k_\sigma \sum_{i=1}^{N} \frac{p_i^{\nu}}{p_i \cdot k} J_i^{\mu\rho} \frac{\partial}{\partial p_{i\sigma}} \mathcal{A}(\{p_i\}) \\
&+ \frac{1}{2} k_\rho \left[\frac{\partial \mathcal{B}^{\mu\nu}}{\partial k_\rho} - \frac{\partial \mathcal{B}^{\rho\nu}}{\partial k_\mu} \right] (\{p_i\}; 0) + \mathcal{O}\left(k^2\right).
\end{aligned} \tag{124}$$

The first line of this equation already shows that the next-to-soft term is universal and proves the corresponding term in (117), while the next-to-next-to-soft $\mathcal{O}(k)$ terms in the second and third lines still contain the anti-symmetric first-order derivatives of the non-singular part $\mathcal{B}^{\mu\nu}$ of the radiative amplitude.

该方程第一行已经表明次软项是普适的，证明了式 (117) 中的对应项，而第二、三行的次次软 $\mathcal{O}(k)$ 项仍包含辐射振幅非奇异部分 $\mathcal{B}^{\mu\nu}$ 的反对称一阶导数。

It is worth noting that the previous equation can be adapted to gauge boson rather than graviton emission by simply replacing $p_i^{\nu} \to t_i^a$ and removing the index ν on $\mathcal{A}_{\text{rad}}^{\mu\nu}$ and $\mathcal{B}^{\mu\nu}$, since the derivation goes through for non-abelian gauge invariance under shifts

值得注意的是，前文的方程可以稍加修改后应用于规范玻色子而非引力子辐射: 只需替换 $p_i^{\nu} \to t_i^a$ ，并去掉 $\mathcal{A}_{\text{rad}}^{\mu\nu}$ 和 $\mathcal{B}^{\mu\nu}$ 上的指标 ν ，因为对位移下的非阿贝尔规范不变性而言，该推导依然成立

$$\varepsilon_\mu(k) \to \varepsilon_\mu(k) + k_\mu \alpha(\{p_i\}; k) \tag{125}$$

of the emitted soft gauge boson polarization vector. The key point about gravitation is that the radiative amplitude also satisfies $k_\nu \mathcal{A}_{\text{rad}}^{\mu\nu}(\{p_i\}; k) = 0$, which can be verified for (124), but has not yet been used explicitly. This fixes the so far undetermined first derivatives in terms of the non-radiative amplitude:

发射出的软规范玻色子极化矢量。引力相关的核心要点是，辐射振幅同样满足 $k_\upsilon \mathcal{A}^{\mu\upsilon}_{\text{rad}}\left(\{p_i\};k\right) = 0$ ，这一点可以对式 (124) 进行验证，不过尚未得到明确应用。该条件可以用非辐射振幅将迄今为止尚未确定的一阶导数固定下来:

$$\left[\frac{\partial \mathcal{B}^{\mu\upsilon}}{\partial k_\rho} - \frac{\partial \mathcal{B}^{\rho\upsilon}}{\partial k_\mu}\right](\{p_i\};0) = \sum_{i=1}^{N} J_i^{\rho\mu} \frac{\partial}{\partial p_{i\upsilon}} \mathcal{A}\left(\{p_i\}\right). \tag{126}$$

Substituting into (124) yields the soft theorem (117) including three universal terms, that is, terms that can be expressed in terms of (derivatives of) the non-radiative amplitude. The authors of [16] further checked that there are not enough constraints from gauge invariance to determine all second derivatives of $\mathcal{B}^{\mu\nu}$; hence, the next $\mathcal{O}\left(k^2\right)$ term in the soft expansion is no longer universal.

代入 (124) 即可得到包含三个普适项的软定理 (117)，这些项都可以用非辐射振幅 (及其导数) 表示。文献 [16] 的作者进一步检查发现，规范不变性提供的约束不足以确定 $\mathcal{B}^{\mu\nu}$ 的所有二阶导数；因此，软展开中的下一个 $\mathcal{O}\left(k^2\right)$ 项不再具有普适性。

The diagrammatic derivation of the soft theorem is remarkably simple on the one hand and highlights the crucial role of gauge invariance. On the other hand, the number of universal terms appears from a rather technical argument, and the derivation provides no physical understanding of why the derivative expansion organizes itself such that the orbital angular momentum operator arises.

一方面，软定理的图解推导非常简洁，还凸显了规范不变性的关键作用。另一方面，普适项的数量是通过相当技术性的论证得出的，该推导也没有从物理层面解释，为什么导数展开会自然呈现出轨道角动量算符。

Soft Theorem from SCET Gravity

来自 SCET 引力的软定理

The soft-collinear effective Lagrangian by construction allows one to generate from its Feynman rules the soft and collinear limits of an amplitude to a desired accuracy in the soft and collinear expansion and the loop expansion. However, it organizes the derivation of the soft theorem in a different form from the above, since the soft factors arise entirely from the emission from external legs. For the case of gauge theory, this was noted first in [7]. [16]

通过构造，软共线有效拉格朗日量可以从其费曼规则出发，在软展开、共线展开与圈展开中，按所需精度给出振幅的软极限与共线极限。但它对软定理的推导采用了与上文不同的组织形式，因为软因子完全来自外腿的辐射。规范场论中的这一结论最早由文献 [7] 给出。[16]

To begin, it is instructive to formulate Weinberg's leading eikonal graviton emission amplitude (2) in the EFT language by recalling from (109) and (99) that an energetic scalar with its large momentum p_i^μ directed along the light-like vector n_{i-}^μ interacts with a soft graviton through the effective Lagrangian [17]

首先，我们用有效场论语言表述温伯格的领头阶艾克诺引力子辐射振幅 (2): 根据 (109) 和 (99)，大动量为 p_i^μ 、类光方向为 n_{i-}^μ 的高能标量通过有效拉格朗日量 [17] 与软引力子相互作用

$$\mathcal{L}_i^{(0)} = \left[n_{i+}\partial\chi_{c_i}\right]^\dagger \left[-\frac{\kappa}{4} n_{i-}^\mu n_{i-}^\nu s_{\mu\nu}(x_{i-})\, i n_{i+}\partial\right] \chi_{c_i}. \tag{127}$$

The structure of (2) is already manifest in this Lagrangian, which couples the energetic particle to the soft gauge field and graviton only proportionally to the large momentum $p_i^\mu \propto n_{i-}^\mu$. The content of the leading term in the soft theorem can now be stated in operatorial form as

振幅 (2) 的结构已经在该拉格朗日量中清晰体现: 高能粒子仅与大动量 $p_i^\mu \propto n_{i-}^\mu$ 成正比地耦合软规范场和引力子。软定理领头项的内容现在可以用算符形式表述为

$$\sum_i i \int d^4x T\left\{\chi_{c_i}^\dagger(0), \mathcal{L}_i^{(0)}(x)\right\}\Big|_{\text{tree}}, \tag{128}$$

where the sum over i runs over the energetic particles created in the hard process. The entire hard, non-radiative N -particle scattering process is sourced by a gauge-invariant product of $N\chi_{c_i}^\dagger$ fields [18] as described in section "Sources and Hard Matching." Contracting $\chi_{c_i}^\dagger(0)$ with $\chi_{c_i}(x)$ in $\mathcal{L}_i^{(0)}$ to form the collinear matter propagator i/p^2 , and taking the matrix element with N matter particles and a soft graviton, immediately yields the amplitude (2). At this point, it is essential that soft gauge bosons and gravitons cannot be emitted directly from the hard vertex at this order in the soft expansion, since there are no source operators containing soft fields that would be invariant under the soft gauge symmetry. The entire radiative amplitude originates from the time-ordered product with the universal Lagrangian interaction. This guarantees the universality of the soft theorem, that is, its form is independent of the non-radiative, hard process. Briefly,

其中对 i 的求和遍历硬过程中产生的高能粒子。整个不辐射的硬 N 粒子散射过程由 $N\chi_{c_i}^\dagger$ 场构成的规范不变乘积 [18] 产生，正如章节“源与硬匹配”所述。将 $\chi_{c_i}^\dagger(0)$ 与 $\mathcal{L}_i^{(0)}$ 中的 $\chi_{c_i}(x)$ 缩并得到共线物质传播子 i/p^2 ，再取 N 个物质粒子和一个软引力子的矩阵元，可直接得到振幅 (2)。在此需要注意，软展开的该阶下，软规范玻色子与引力子无法直接从硬顶点发射，因为不存在满足软规范对称性不变性且包含软场的源算符。全部辐射振幅都来自与普遍拉格朗日相互作用的时序乘积。这保证了软定理的普遍性，即其形式与不辐射的硬过程无关。简言之，

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[16] This statement holds in the position-space SCET framework. For a discussion of the LBK theorem in the so-called label formalism, see [30].

[16] 该结论在位置空间 SCET 框架中成立。关于所谓标号形式体系下的 LBK 定理，参见文献 [30]。

[17] In the following, we drop the hats of the collinear fields. Furthermore, we consider a complex scalar field. This has the advantage that in the convention where all external particles are outgoing, the N -jet operator contains one field $\chi_c^\dagger$ for each particle. Thus, one can always identify $\chi_c^\dagger$ in the Lagrangian as the outgoing field, simplifying the following derivation.

[17] 在下文中，我们省略共线场的帽子符号。此外我们考虑复标量场，这样做的好处是: 在所有外粒子均为出射的约定下，N 喷注算符对每个粒子都对应一个场 $\chi_c^\dagger$。因此我们总能将拉格朗日量中的 $\chi_c^\dagger$ 识别为出射场，简化后续推导。

[18] Employing the convention that all particles are outgoing

[18] 采用所有粒子均为出射的约定

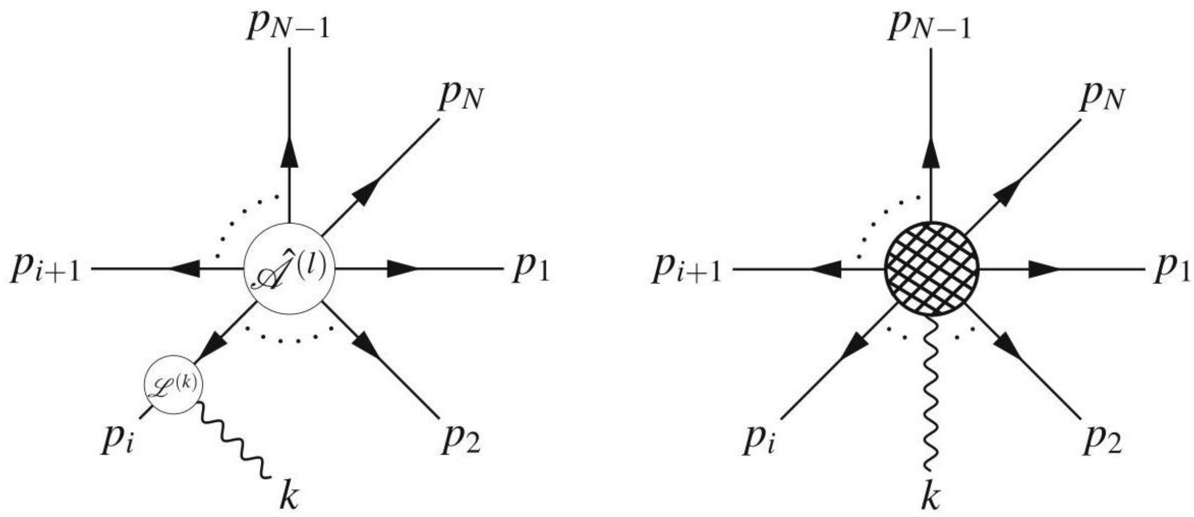

Fig. 6 Possible contributions to the radiative amplitude in SCET. The first diagram represents the time-ordered product of the current and the Lagrangian. Both current and Lagrangian can be suppressed by a single power of λ , or there can be a contribution from the λ^2 -suppressed Lagrangian together with the leading-power current. There are no soft building blocks at order λ^2 and λ^4 ; hence, the second diagram does not contribute to the amplitude in SCET

图 6 SCET 中辐射振幅的可能贡献。第一张图代表流与拉格朗日量的时序乘积。流和拉格朗日量都可以被 λ 压低一个幂次，也可以由 λ^2 压低的拉格朗日量结合领头幂次流给出贡献。由于不存在 λ^2 阶和 λ^4 阶的软构造块，因此第二张图对 SCET 中的振幅没有贡献

Lagrangian insertions ⇒ universal terms, while the source operators in the effective theory have process-dependent matching coefficients to the full theory. Thus, when there is a source operator with N collinear fields and a soft graviton, it will contribute a non-universal term to the soft expansion of the radiative amplitude.

拉格朗日量插入项 ⇒ 都是普适项，而有效理论中的源算符连接到完整理论时带有过程依赖的匹配系数。因此，若源算符含有 N 个共线场和一个软引力子，它会给辐射振幅的软展开贡献一个非普适项。

Unlike for amplitudes, in soft-collinear effective theory, the separation of the amplitude into emission from the external leg and the hard vertex (see Fig. 6) is gauge-invariant and corresponds to the separation into gauge-invariant Lagrangian insertions and gauge-invariant source operators with soft graviton fields. The number of universal terms in the soft expansion follows from this observation without the need for an explicit computation, since, as noted in section ”Gauge-Invariant Building Blocks,” the leading gauge-invariant soft building block for gravitons is the Riemann tensor, which scales as $\mathcal{O}\left(\lambda^6\right)$, corresponding to a third-order correction in the soft expansion. It is this simple consequence of the soft gauge symmetry of the effective Lagrangian, which implies that there is some form of universal soft theorem including a universal next-to-next-to-soft term for gravity, as any soft emission up these orders must arise from universal Lagrangian terms, independent of the source for the energetic particles. The additional universal term compared to gauge boson emission follows from the observation that the leading soft building block for soft gauge fields is the field strength tensor $F^s_{\mu\nu}$, which scales as $\mathcal{O}\left(\lambda^4\right)$; hence, there are non-universal contributions already at the next-to-next-to-soft order.

与振幅不同，在软共线有效理论中，将振幅分解为外腿辐射和硬顶点 (见图 6) 是规范不变的，且对应将其分解为规范不变拉氏量插入项和携带软引力子场的规范不变源算符。软展开中普适项的数量可由该观测结论直接得到，无需显式计算: 正如“规范不变构造模块”一节所述，引力子的领头阶规范不变软构造模块是黎曼张量，其标度为 $\mathcal{O}\left(\lambda^6\right)$，对应软展开中的三阶修正。这正是有效拉氏量软规范对称性的简单推论，它表明引力存在某种形式的普适软定理，其中包含普适的次次软项——因为到这些阶的任何软辐射都必须来自普适拉氏量项，与高能粒子的源无关。与规范玻色子辐射相比，额外普适项的来源是: 软规范场的领头阶软构造模块是场强张量 $F^s_{\mu\nu}$，其标度为 $\mathcal{O}\left(\lambda^4\right)$，因此次次软阶 already 存在非普适贡献。

This answers the first question. To gain a better understanding of the detailed form of the soft theorem, in particular the appearance of the angular momentum operator in the sub-leading terms, one must investigate the structure of the effective Lagrangian. As before, the specific case of scalar matter will be considered. Since the soft theorem is a statement about tree-level, single graviton emission, two simplifications can be made in the general soft-collinear gravity Lagrangian:

这就回答了第一个问题。为了更清楚地理解软定理的具体形式，尤其是次领头项中角动量算符的出现，我们必须研究有效拉氏量的结构。和之前一样，本文将讨论标量物质的特殊情形。由于软定理是关于树图阶单引力子辐射的结论，我们可以对一般软共线引力拉氏量做两处简化:

- The collinear graviton field can be set to zero, since there are no internal or external collinear graviton lines.

- 共线引力子场可设为零，因为不存在内壳或外壳共线引力子线。

- Only linear interactions in the soft graviton field need to be retained.

- 只需要保留软引力子场的线性相互作用。

In the remainder of this subsection, the basic ideas will be explained for the next-to-soft term, which counts as $\mathcal{O}\left(\lambda^2\right)$. For the derivation of the sub-sub-leading soft term and more technical detail, we refer to [10]. This reference also discusses the case of non-abelian gauge theory, including the case of matter with spin $\frac{1}{2}$ and 1, which demonstrates how the spin term $\sum^{\mu\nu}$ in $J^{\mu\nu}$ is encoded in the effective Lagrangian.

在本小节余下部分，我们将解释次软项的基本思想，其计数为 $\mathcal{O}\left(\lambda^2\right)$。关于次次领头软项的推导和更多技术细节，请参考文献 [10]。该文献还讨论了非阿贝尔规范理论的情形，包括自旋为 $\frac{1}{2}$ 和 1 的物质情形，说明了 $J^{\mu\nu}$ 中的自旋项 $\sum^{\mu\nu}$ 如何编码在有效拉氏量中。

The soft-collinear effective Lagrangian for the complex gauge-invariant scalar field χ_c up to $\mathcal{O}\left(\lambda^2\right)$, with simplifications as stated above already applied, can be expressed in terms of the energy-momentum tensor (dropping the index i when referring to the Lagrangian, since its form is the same for all i)

针对复规范不变标量场 χ_c 、精度到 $\mathcal{O}\left(\lambda^2\right)$ 的软共线有效拉氏量，在应用上述简化后，可以用能量动量张量表示 (提及拉氏量时我们省略下标 i ，因为对所有 i 其形式都相同)

$$T^{\mu\upsilon} = [\partial^\mu \chi_c]^\dagger \partial^\upsilon \chi_c + [\partial^\upsilon \chi_c]^\dagger \partial^\mu \chi_c - \eta^{\mu\upsilon} [\partial_\alpha \chi_c]^\dagger \partial^\alpha \chi_c \tag{129}$$

as

$$\mathcal{L} = \frac{1}{2}[n_+ \partial \chi_c]^\dagger n_- \partial \chi_c + \frac{1}{2}[n_- \partial \chi_c]^\dagger n_+ \partial \chi_c + \left[\partial_{\mu_\perp} \chi_c^\dagger\right] \partial^{\mu_\perp} \chi_c \tag{130}$$

$$-\frac{\kappa}{4} s_{-\mu} T^{\mu+} - \frac{\kappa}{4}\left[\partial_{[\mu} s_{\upsilon]-}\right](x - x_-)^\mu T^{\upsilon+} - \frac{1}{8} x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-} T^{++} + \mathcal{O}\left(x^3\right).$$

When no space-time argument is specified, all soft fields are evaluated at x_-^μ after derivatives are taken. In this form, the coupling of $s_{\mu-}$ to the energy-momentum tensor $T^{\mu\upsilon}$ becomes transparent, as well as the coupling of its derivative $\partial_{[\alpha} s_{\beta]-}$, which is an independent gauge field in the effective theory, to the angular momentum density

当未指定时空自变量时，所有软场都在求导后于 x_-^μ 处取值。在该形式下，$s_{\mu-}$ 与能量动量张量 $T^{\mu\upsilon}$ 的耦合，以及其导数 $\partial_{[\alpha} s_{\beta]-}$ (它是有效理论中独立的规范场) 与角动量密度的耦合都变得清晰明了。

$$\mathcal{J}^{\alpha\beta\mu} = (x - x_-)^\alpha T^{\beta\mu} - (x - x_-)^\beta T^{\alpha\mu}. \tag{131}$$

The Lagrangian (130) is not yet homogeneous in λ . The λ -expansion is made manifest by decomposing the contraction of the indices μ, ν in light cone components. Then

拉氏量 (130) 还不是按 λ 齐次分类的。我们可以通过将指标 μ, ν 的缩并分解为光锥分量来得到显式的 λ 展开，此时

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{O}\left(\lambda^3\right), \tag{132}$$

where

其中

$$\mathcal{L}^{(0)} = \frac{1}{2}[n_+ \partial \chi_c]^\dagger n_- \partial \chi_c + \frac{1}{2}[n_- \partial \chi_c]^\dagger n_+ \partial \chi_c + [\partial_{\mu_\perp} \chi_c]^\dagger \partial^{\mu_\perp} \chi_c - \frac{\kappa}{8} s_{--} T^{++},$$

(133)

$$\mathcal{L}^{(1)} = -\frac{\kappa}{4} s_{-\mu_\perp} T^{\mu_\perp +} - \frac{\kappa}{8} \left[\partial_{[\mu} s_{-]-}\right] x_\perp^\mu T^{++}, \tag{134}$$

$$\mathcal{L}^{(2)} = -\frac{\kappa}{8} s_{+-} T^{+-} - \frac{\kappa}{4} \left[\partial_{[\mu_\perp} s_{\nu_\perp]-}\right] x_\perp^\mu T^{\nu_\perp +} - \frac{\kappa}{16} \left[\partial_{[+} s_{-]-}\right] n_- x T^{++}$$

$$-\frac{1}{8} x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-} T^{++}. \tag{135}$$

The source operator $\widehat{\mathcal{A}}$ that represents the hard scattering process also needs to be expanded in λ :

代表硬散射过程的源算符 $\widehat{\mathcal{A}}$ 也需要按 λ 展开:

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^{(0)} + \widehat{\mathcal{A}}^{(1)} + \widehat{\mathcal{A}}^{(2)} + \mathcal{O}\left(\lambda^3 \widehat{\mathcal{A}}^{(0)}\right). \tag{136}$$

The $\widehat{\mathcal{A}}^{(n)}$ up to this order are operators containing $i\partial_{\perp i}^{\mu_1} \dots i\partial_{\perp i}^{\mu_n} \chi_{c_i}^\dagger$, which corresponds to the expansion of the non-radiative momentum space hard scattering amplitude in the transverse momentum $p_{i\perp}^\mu$ of particle i . The operatorial version of the soft theorem then amounts to the expansion in λ of the right-hand side of

到该阶为止的 $\widehat{\mathcal{A}}^{(n)}$ 是包含 $i\partial_{\perp i}^{\mu_1} \dots i\partial_{\perp i}^{\mu_n} \chi_{c_i}^\dagger$ 的算符，对应非辐射动量空间硬散射振幅按粒子 i 横动量 $p_{i\perp}^\mu$ 展开。软定理的算符形式等价于对右侧按 λ 展开

$$\widehat{\mathcal{A}}_{\text{rad}} = i \int d^4x T\{\widehat{\mathcal{A}}, \mathcal{L}\} \tag{137}$$

in the tree-level approximation and under the assumption that the operator is evaluated in the state $\left\langle \{p_i\}; k \middle| \widehat{\mathcal{A}}_{\text{rad}} \middle| 0 \right\rangle$.

在树图近似下，且假设该算符在态 $\left\langle \{p_i\}; k \middle| \widehat{\mathcal{A}}_{\text{rad}} \middle| 0 \right\rangle$ 下求值。

It is always possible to define the N light cone vectors n_{i-}^μ such that the external energetic matter particle momenta of the radiative amplitude satisfy $p_{i\perp}^\mu = 0$ for all i with respect to their light cone vectors. This choice simplifies the calculations considerably. Many terms vanish, which would otherwise contribute to restoring Lorentz-invariant scalar products of the external momenta. These scalar products do not depend on the specific choice of the reference vectors that break manifest Lorentz invariance. In particular, one finds that for $n > 0$

我们总能定义 N 光锥矢量 n_{i-}^μ ，使得辐射振幅的外部高能物质粒子动量对所有 i 都满足关于自身光锥矢量的条件 $p_{i\perp}^\mu = 0$ 。该选择大幅简化了计算: 许多原本会贡献恢复外动量洛伦兹不变标量积的项都变为零，而这些标量积本身不依赖破坏显洛伦兹不变性的参考矢量的特定选取。特别地，我们发现对于 $n > 0$

$$T\left\{\widehat{\mathcal{A}}^{(n)}, \mathcal{L}^{(0)}\right\}, T\left\{\widehat{\mathcal{A}}^{(0)}, \mathcal{L}^{(1)}\right\} \tag{138}$$

evaluate to zero on the right-hand side of (137), leaving [19]

在 (137) 右侧求值为零，得到 [19]

$$\widehat{\mathcal{A}}^{(2)}_{\text{rad}} \triangleq i\int d^4x T\left\{\widehat{\mathcal{A}}^{(1)}, \mathcal{L}^{(1)}\right\} + i\int d^4x T\left\{\widehat{\mathcal{A}}^{(0)}, \mathcal{L}^{(2)}\right\} \tag{139}$$

for the next-to-soft term.

对应次软项。

Examining (134) and (135), one finds that the Riemann tensor term in the last line of (135) does not contribute to the time-ordered product with $\widehat{\mathcal{A}}^{(0)}$ in the frame

考察 (134) 和 (135) 可以发现，(135) 最后一行的黎曼张量项在该坐标系下与 $\widehat{\mathcal{A}}^{(0)}$ 的时序对易子中没有贡献

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[19] The symbol ≙ means "equal up to terms that do not contribute to the specific tree-level matrix elements considered here, i.e., with single soft emission, no collinear emissions, and ⊥ -component of the external collinear particle momenta set to zero."

[19] 符号 ≙ 表示"等于不贡献本文所讨论特定树图矩阵元的项，即仅含单次软辐射、无共线辐射、且外部共线粒子动量的 ⊥ 分量设为零的项"。

$p^{\mu}_{i\perp} = 0$, while the remaining terms and $\mathcal{L}^{(1)}$ can be rewritten after integrating by parts into

$p^{\mu}_{i\perp} = 0$，分部积分后剩余项和 $\mathcal{L}^{(1)}$ 可重写为

$$\mathcal{L}^{(1)} \triangleq \frac{\kappa}{2}\left[\partial_\mu s_{\nu-}\right]\chi_c^\dagger \overleftarrow{L}^{\mu\nu}_{+\perp} n_+\partial\chi_c, \tag{140}$$

$$\mathcal{L}^{(2)} \triangleq \frac{\kappa}{2}\left[\partial_\mu s_{\nu-}\right]\chi_c^\dagger \overleftarrow{L}^{\mu\nu}_{+-} n_+\partial\chi_c, \tag{141}$$

where

其中

$$L^{\mu\nu} = x^{[\mu]}\partial^{\nu]} = \frac{1}{2}x^{[\mu n_-^{\nu}]}n_+\cdot\partial + \ldots = \underbrace{\frac{1}{4}n_-\cdot x n_+\cdot\partial n_+^{[\mu}n_-^{\nu]}}_{L^{\mu\nu}_{+-}} + \underbrace{\frac{1}{2}x_\perp^{[\mu}n_-^{\nu]}n_+\cdot\partial}_{L^{\mu\nu}_{\perp+}}$$

(142)

is the orbital angular momentum operator in position space and the dots after the first equality represent terms that vanish in the $p^{\mu}_{\perp} = 0$ frame. Both time-ordered products in (139) can now be combined into

是位置空间的轨道角动量算符，第一个等号后的点代表在 $p^{\mu}_{\perp} = 0$ 坐标系下为零的项。(139) 中的两个时序乘积现在可以合并为

$$\widehat{\mathcal{A}}^{(2)}_{\text{rad}} = \int d^4x T\left\{\widehat{\mathcal{A}}, \frac{\kappa}{2}\chi_c^{\dagger}\overleftarrow{L}^{\mu\upsilon}\left[\partial_\mu s_{\upsilon-}\right] in_+\partial\chi_c\right\}, \tag{143}$$

which is precisely the next-to-soft term in (117). This answers the second question: the angular momentum operator is seen to appear since the soft-collinear expansion of the Lagrangian together with the light cone multipole expansion of soft interactions with energetic particles naturally provides the right structures. At the next-to-soft order, the appearance of the angular momentum is already evident in (130) and related to the soft gauge symmetry of the effective Lagrangian.

这正好是 (117) 中的次软项。这回答了第二个问题: 角动量算符之所以出现，是因为拉格朗日的软共线展开，加上高能粒子软相互作用的光锥多极展开，自然给出了正确的结构。在次软阶，角动量的出现在 (130) 中就已经很明显，且和有效拉格朗日的软规范对称性有关。

This observation provides a deeper understanding of the gravitational soft theorem. Unlike the gauge theory case, where the next-to-soft term in the LBK theorem can be shown to be related to the soft field strength tensor $F^s_{\mu\nu}$ [10] and is therefore gauge-invariant without requiring a conserved quantity in the scattering process, the next-to-soft term in gravity takes the form of an eikonal term, just as the leading term. The leading term

这一观测让我们对引力软定理有了更深刻的理解。规范理论中，LBK 定理的次软项可以证明和软场强张量 $F^s_{\mu\nu}$ [10] 相关，因此不需要散射过程存在守恒量就是规范不变的；与之不同，引力的次软项和领头项一样，都取程函项的形式。领头项

$$\varepsilon_\mu - p^\mu \frac{n+p}{p\cdot k} \tag{144}$$

is related to the coupling of the first soft gauge field $s_{-\mu}$ to the energy-momentum tensor $T^{\mu+}$ in (130). It is gauge-invariant due to the conservation of momentum. The first sub-leading order in the soft theorem,

和 (130) 中第一个软规范场 $s_{-\mu}$ 与能量动量张量 $T^{\mu+}$ 的耦合有关。它因动量守恒而具有规范不变性。软定理中的第一个次领头阶，

$$k_\rho \varepsilon_{\mu-} J^{\rho\mu} \frac{n+p}{p\cdot k}, \tag{145}$$

is generated by the next term in (130). It involves the second soft gauge field, $\partial_{[\mu}s_{\upsilon]-} \equiv \partial_\mu s_{\upsilon-} - \partial_\upsilon s_{\mu-}$ (or $[\Omega_-]_{\mu\upsilon}$; see (96)), which couples to the energy-momentum density. The next-to-soft term for gravity is only gauge-invariant once angular momentum conservation of the scattering process is imposed. Hence, it is the twofold emergent soft gauge symmetry that implies two eikonal terms in the soft theorem, consistent with the covariant derivative

由式 (130) 的下一项产生。它包含第二个软规范场 $\partial_{[\mu}s_{\upsilon]-} \equiv \partial_\mu s_{\upsilon-} - \partial_\upsilon s_{\mu-}$ (或 $[\Omega_-]_{\mu\upsilon}$ ，见式 (96))，该规范场与能量动量密度耦合。仅当满足散射过程角动量守恒时，引力的次软项才是规范不变的。因此，二重涌现软规范对称性意味着软定理中存在两个程函项，与协变导数一致

$$n_- D_s = \partial_- - \frac{\kappa}{2} s_{-\mu}\partial^\mu - \frac{\kappa}{4}\partial_{[\mu}s_{\upsilon]-}J^{\mu\upsilon} + \dots . \tag{146}$$

Finally, the Riemann tensor terms, which are present in the sub-leading soft-collinear interaction Lagrangians $\mathcal{L}^{(2)}, \mathcal{L}^{(3)}$, and $\mathcal{L}^{(4)}$, generate the next-to-next-to-soft term [10]

最后，次领头软共线相互作用拉格朗日量 $\mathcal{L}^{(2)}, \mathcal{L}^{(3)}$ 和 $\mathcal{L}^{(4)}$ 中存在的黎曼张量项产生次次软项 [10]

$$\frac{1}{2}\varepsilon_{\mu\nu}k_\rho k_\sigma J^{\rho\mu}\frac{J^{\sigma\nu}}{p\cdot k} \tag{147}$$

in (117). Here, one factor of $J^{\mu\upsilon}$ is related to the charge of one of the soft gauge symmetries, while the other arises from the kinematic multipole expansion. Originating from the Riemann tensor, this term is manifestly gauge-invariant without requiring further (non-existent) conserved charges. Thus, despite appearance, the next-to-next-to soft term in gravity has the same physical origin as the next-to-soft term in the LBK theorem, while the soft and next-to-soft term should be viewed as the gravitational analogues of the familiar eikonal factor in gauge theories.

在式 (117) 中。此处，一个 $J^{\mu\upsilon}$ 因子对应其中一个软规范对称性的荷，另一个则来自动力学多极展开。该项起源于黎曼张量，无需额外 (不存在的) 守恒荷即可明显满足规范不变性。因此，尽管看似不同，引力中的次次软项与 LBK 定理中的次软项物理来源相同，而软项和次软项应视作规范理论中常见程函因子的引力类比。

Loop Corrections to the Soft Theorem

软定理的圈修正

While the soft theorem is agnostic about the details of the hard non-radiative amplitude that produces the N energetic particles (in widely separated directions), it holds only at tree level in the interactions of the energetic particles with soft gravitons. Soft-collinear gravity generates the soft and collinear loops in gravitational scattering to any order, so it is natural to ask whether it provides insight on the loop corrections to the soft theorem. In this subsection, we will show that:

软定理不依赖产生 N 高能粒子 (朝向分离很远) 的硬非辐射振幅的细节，但它仅在高能粒子与软引力子相互作用的树图阶成立。软共线引力可将引力散射中的软圈与共线圈生成到任意阶，因此自然可以探究它是否能为软定理的圈修正提供启发。在本小节中，我们将证明:

The leading soft factor is not modified by loop effects. The sub-leading factor is only corrected by one-loop, and the sub-sub-leading factor is only modified by one- and two-loop contributions. Higher-order loop corrections do not affect the gravitational soft theorem.

领头软因子不受圈效应修正。次领头软因子仅接受单圈修正，次次领头软因子仅被单圈和双圈贡献修正。高阶圈修正不影响引力软定理。

This should be compared to non-abelian gauge theory, where already the leading term receives loop corrections of any order. The above statement was already made in [15], but the reasoning from EFT provided in [10] and below is somewhat different and relies only on (1) power counting, (2) the eikonal identity, and (3) the necessity to form a soft invariant from a given soft loop integral, as otherwise the integral is scaleless and vanishes. It should be noted that the radiative amplitude is infrared-divergent at loop level and a regularization is needed. The above statement holds when singularities are regulated dimensionally in $d = 4-2\varepsilon$ dimensions.

这一点需要与非阿贝尔规范理论对比，非阿贝尔规范理论中领头项已经会接受任意阶的圈修正。上述结论最早在文献 [15] 中提出，但文献 [10] 以及下文给出的有效场论推导思路有所不同，且仅依赖三点:(1) 幂次计数，(2) 程函恒等式，(3) 从给定软圈积分构造软不变量的必要性——否则该积分无标度且等于零。需要注意，辐射振幅在圈阶存在红外发散，需要正则化。上述结论在 $d = 4-2\varepsilon$ 维对奇点进行维度正则化时成立。

In SCET, loop contributions arise from three different loop momentum regions, the hard, the collinear, and the soft region, the latter two corresponding to collinear and soft modes in the effective theory. The hard modes are integrated out; thus, the contributions of the hard loops are inside the matching coefficients $\widetilde{C}^X(t_i)$ and consequently part of the non-radiative amplitudes $\langle p_1, \dots, p_N | \mathcal{J}(0) | 0\rangle$ defined in section "Sources and Hard Matching." Hence, hard loops never affect the soft theorem directly - they modify the underlying non-radiative process.

在软共线有效理论 (SCET) 中, 圈贡献来自三个不同的圈动量区域: 硬区域、共线区域和软区域, 后两个对应有效理论中的共线模式和软模式。硬模式已被积掉，因此硬圈贡献都包含在匹配系数 $\widetilde{C}^X(t_i)$ 中，属于“源与硬匹配”一节定义的非辐射振幅 $\langle p_1, \dots, p_N | \mathcal{J}(0) | 0\rangle$ 的一部分。因此硬圈从不直接影响软定理，它们只会修正底层的非辐射过程。

Soft-collinear gravity differs from gauge theory in the purely soft and collinear sectors, ultimately due to the dimensionful gravitational coupling:

软共线引力与规范理论在纯软区域和纯共线区域存在差异，这归根结底源于量纲为质量的引力耦合:

(i) There are no collinear singularities. In the purely collinear sector, that is, in the Lagrangian terms containing only collinear but no soft fields, there are no leading-power interactions. The λ -expansion corresponds to the weak-field expansion, and the first collinear interaction appears in $\mathcal{O}(\lambda)$. Purely collinear gravity is an expansion in collinear momenta $p_\perp \sim \lambda$.

(i) 不存在共线奇点。在纯共线区，也就是拉格朗日中仅含共线场不含软场的项中，没有领头幂次相互作用。λ 展开对应弱场展开，第一个共线相互作用出现在 $\mathcal{O}(\lambda)$ 。纯共线引力是对共线动量 $p_\perp \sim \lambda$ 的展开。

(ii) In the purely soft sector, that is, in the Lagrangian terms containing only soft but no collinear fields, there are also no leading-power interactions. Here, the weak-field expansion agrees with the λ^2 -expansion, corresponding to an expansion in soft momenta $k \sim \lambda^2$. Purely soft interaction vertices thus start at $\mathcal{O}(\lambda^2)$.

(ii) 在纯软区，也就是拉格朗日中仅含软场不含共线场的项中，同样不存在领头幂次相互作用。此处弱场展开与 λ^2 展开一致，对应对软动量 $k \sim \lambda^2$ 的展开。因此纯软相互作用顶点从 $\mathcal{O}(\lambda^2)$ 阶开始出现。

Hence, whenever a purely collinear or a purely soft interaction takes place, the contribution is already suppressed by at least one order of λ or λ^2 , respectively. In gravity, only soft-collinear interactions of the eikonal type exist at leading power.

因此，只要发生纯共线相互作用或纯软相互作用，其贡献就会分别至少被一个幂次的 λ 或 λ^2 压低。在引力中，仅程函类型的软共线相互作用出现在领头幂次。

Consider first adding only collinear loops to the single soft emission amplitude. At the one-loop level, there exist two possibilities, shown in Fig. 7. First, a collinear graviton field $\mathfrak{h}_{\perp\perp}$ in a given collinear direction n_{i-}^{μ} can be added to the source, which comes with an extra power of λ for the hard amplitude. The collinear graviton must be attached to the external leg i through a collinear interaction vertex, which costs at least another power of λ , resulting in at least $\mathcal{O}\left(\lambda^2\right)$ suppression of the collinear loop. Second, the collinear loop correction can be on the external leg i only with no connection to the hard vertex, in which case one needs either two purely collinear vertices of at least $\mathcal{O}(\lambda)$ or a four-point vertex, which is already $\mathcal{O}\left(\lambda^2\right)$. By the same argument, adding any further collinear loop incurs at least another factor $\mathcal{O}\left(\lambda^2\right)$ per loop. It follows that the leading term in the soft theorem can never be corrected, while a collinear n -loop correction can contribute only at the n th sub-leading order to the soft theorem, consistent with the claim. Note that the argument holds for every collinear direction i separately, since the collinear modes from different directions cannot interact other than through soft modes, as evident from (41), which implies soft loops.

首先考虑仅对单个软辐射振幅添加共线圈。单圈层面存在两种可能，如图 7 所示。第一，给定共线方向 n_{i-}^{μ} 中的共线引力子场 $\mathfrak{h}_{\perp\perp}$ 可以添加到源中，这会给硬振幅带来额外一次 λ 的幂次。共线引力子必须通过共线相互作用顶点连接到外线 i ，这至少需要再一次 λ 的幂次，因此共线圈至少存在 $\mathcal{O}\left(\lambda^2\right)$ 压低。第二，共线圈修正可以仅出现在外线 i 上，不连接硬顶点，这种情况下需要至少两个幂次为 $\mathcal{O}(\lambda)$ 的纯共线顶点，或者一个已经是 $\mathcal{O}\left(\lambda^2\right)$ 幂次的四点顶点。同理，每额外添加一个共线圈，至少会再带来一个因子 $\mathcal{O}\left(\lambda^2\right)$。由此可得，软定理的领头阶永远不会被修正，而共线 n 圈修正对软定理的贡献仅出现在次领头阶的 n 阶，与我们的结论一致。注意该论证对每个共线方向 i 分别成立，因为来自不同方向的共线模只能通过软模发生相互作用，这一点从 (41) 式可以看出，它暗示了软圈的存在。

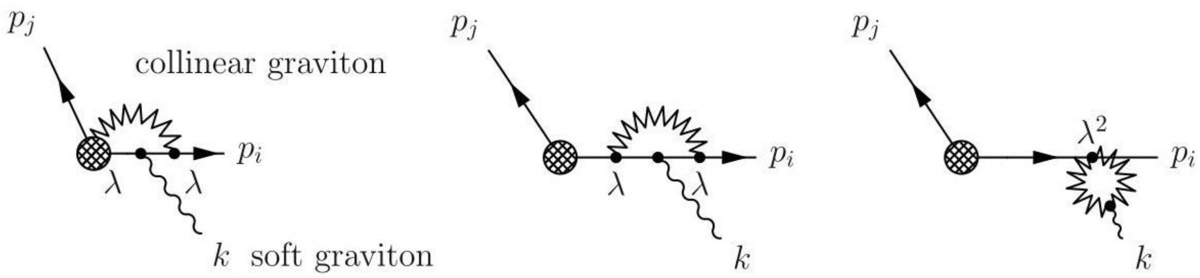

Fig. 7 Collinear one-loop corrections. Left: Collinear graviton from the source in one of the N given directions. Middle and right: Collinear loop corrections to an external leg. Two further diagrams similar to the first two but with the soft graviton attached to the collinear graviton line are not shown

图 7 共线单圈修正。左图: 来自源、处于 N 个给定方向之一的共线引力子。中图和右图: 外线的共线圈修正。另外两个与前两个类似、但软引力子连接在共线引力子线上的图未画出

The case of soft loops is less straightforward. Since there are no hard vertices with soft fields up to sub-sub-leading order, all soft loops to this order must be built from Lagrangian interactions. We first consider the case that all loops are soft and none is collinear.

软圈的情况没那么直接。由于直到次次领头阶都不存在含软场的硬顶点，该阶的所有软圈都只能由拉格朗日相互作用构造。我们首先考虑所有圈都是软圈、没有共线圈的情况。

Beginning with one-loop soft corrections, the soft loop can connect at most two collinear directions with topologies, as shown in Fig. 8, and a similar set of diagrams, where the soft loop is attached to a single leg only. These contributions vanish unless the external soft graviton is connected to the loop through a purely soft interaction, as shown in the right-most diagram in the figure. [20] The key point is that soft fields are always multipole-expanded in interactions with collinear fields, which implies that for soft momentum p_s , only $(n_{i-} p_s)\, n_{i+}^{\mu}/2$ enters the momentum conservation delta function at a soft-collinear vertex and hence the collinear propagators in direction i . The loop depicted in Fig. 8, with the soft graviton emission removed, is given by the (dimensionally regulated) integral

从单圈软修正开始，软圈最多可以用图 8 所示的拓扑连接两个共线方向，还有一组类似的图，其中软圈仅连接在单个外线上。这些贡献都为零，除非外部软引力子通过纯软相互作用连接到圈，正如图中最右侧的图所示。[20] 关键在于，软场在与共线场相互作用时总是会做多极展开，这意味着对于软动量 p_s ，只有 $(n_{i-} p_s)\, n_{i+}^{\mu}/2$ 进入软共线顶点的动量守恒 δ 函数，进而进入方向 i 上的共线传播子。图 8 所示的圈，去掉软引力子辐射后，由 (维度正则化的) 积分给出

$$I \propto \int \frac{d^d l}{(2\pi)^d} \frac{1}{p_i^2 + n_{i+} p_i n_{i-} l + i0} \frac{1}{p_j^2 - n_{j+} p_j n_{j-} l + i0} \frac{1}{l^2 + i0}, \tag{148}$$

which for on-shell external particles, $p_i^2 = p_j^2 = 0$, simplifies to

对于在壳外部粒子 $p_i^2 = p_j^2 = 0$ ，该式可简化为

$$I \propto \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + i0} \frac{1}{n_{i-} l + i0} \frac{1}{n_{j-} l + i0}. \tag{149}$$

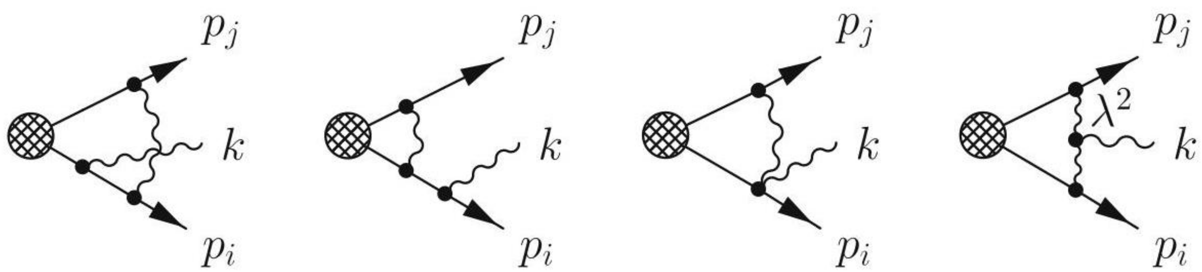

Fig. 8 Diagram classes modifying the soft emission process at the one-loop order. The soft-collinear interactions in the first two diagrams are present at leading power in λ . However, due to multipole expansion, soft-collinear vertices are only sensitive to the $n_{i-} k$ component, and the soft loop vanishes unless a soft scale, provided by the injection of the full soft momentum k , is present. This can only happen by a purely soft interaction vertex. Hence, only the right-most diagram is non-vanishing, but is at least $\mathcal{O}\left(\lambda^2\right)$ suppressed, since it contains a purely soft interaction

图 8 单圈阶修正软辐射过程的图类。前两个图中的软共线相互作用都处于 λ 的领头幂次。然而，由于多极展开，软共线顶点仅对 $n_{i-}k$ 分量敏感，除非注入了携带完整软动量 k 的软标度，否则软圈为零。这只能通过纯软相互作用顶点实现。因此，只有最右侧的图非零，但它至少存在 $\mathcal{O}\left(\lambda^2\right)$ 压低，因为它包含一个纯软相互作用

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[20] See Appendix B of [4] for a similar discussion for soft interactions in gauge theory at next-to-leading power.

[20] 关于规范论中次领头幂次软相互作用的类似讨论，参见文献 [4] 的附录 B。

This integral is evidently scaleless and vanishes. Note that the diagrams of Fig. 8 provide numerators for this integral, which are polynomial in the external and the loop momentum. They have not been written explicitly, since whether an integral is scaleless or not is independent of such numerators.

该积分显然是无标度积分，结果为零。注意图 8 的图为该积分提供了分子，这些分子是外部动量和圈动量的多项式。我们没有将它们显式写出，因为一个积分是否是无标度积分，与这些分子无关。

If one now attaches the external soft graviton to one of the collinear lines (see the first three diagrams in Fig. 8), one can always route the external soft momentum k such that it appears in the eikonal propagators of only one of the legs, say i . In this way, the loop integral (149) may be modified to include eikonal propagators of the form $\left(n_{i-}\left(l+k\right)+i0\right)^{-1}$. Since only the $n_{i-}kn_{i+}^{\mu}/2$ component of the soft momentum can ever appear in the denominator, one cannot form an invariant scalar product containing k with the required soft scaling $\mathcal{O}\left(\lambda^4\right)$ (as $n_{i+}^2 = 0$), and the soft loop integral will remain scaleless and vanishing. In order for soft loops to yield a non-zero contribution, one needs to bring the full external soft momentum k^{μ} into the loop integral. This requires the external soft graviton to couple to the loop through a purely soft interaction (as in the right-most diagram in the figure). Such interaction vertices involve the full momentum conservation delta function and lead to propagators $1/(l+k)^2$, which allows the soft integral to depend on the soft invariant $\left(n_{i-}k\right)\left(n_{j-}k\right)\left(n_{i+}n_{j+}\right)$ and be non-zero. However, by point (ii) above, such a purely soft vertex comes at the cost of power suppression by at least λ^2 . Hence, soft one-loop corrections also cannot affect the leading term in the soft theorem; however, the next-to-soft term can be modified by diagrams with the external soft graviton attached to a purely soft vertex.

如果现在将外部软引力子连接到其中一条共线线上 (参见图 8 中的前三个图)，我们总可以对外部软动量 k 作流道分配，使得它仅出现在其中一条支腿 (例如 i) 的程函传播子中。通过这种方式，圈积分 (149) 可被修改为包含形式为 $\left(n_{i-}\left(l+k\right)+i0\right)^{-1}$ 的程函传播子。由于只有软动量的 $n_{i-}kn_{i+}^{\mu}/2$ 分量能出现在分母中，我们无法构造出包含 k 且满足所需软标度 $\mathcal{O}\left(\lambda^4\right)$ 的不变标量积 (如 $n_{i+}^2 = 0$)，因此软圈积分仍是无标度的，结果为零。要让软圈给出非零贡献，我们需要将完整的外部软动量 k^{μ} 引入圈积分中。这要求外部软引力子通过纯软相互作用耦合到圈 (如图中最右侧的图所示)。这类相互作用顶点包含完整的动量守恒 δ 函数，得到传播子 $1/(l+k)^2$ ，这使得软积分可以依赖于软不变量 $\left(n_{i-}k\right)\left(n_{j-}k\right)\left(n_{i+}n_{j+}\right)$ ，从而结果非零。但根据上述第 (ii) 点，这类纯软顶点至少会带来 λ^2 的幂次压低。因此，软单圈修正也不会影响软定理中的领头项；不过次软项可以被外部软引力子连接到纯软顶点的图修改。

The above argument generalizes to the following all-order statement: In soft loop corrections to the soft theorem, contrary to the tree-level case, the emitted soft graviton must always attach to a purely soft vertex and never directly to any of the energetic particle lines. The reason is that soft-collinear interactions involve the soft field at the multipole-expanded point x_-^μ to any order in the λ -expansion. Hence, if the emitted graviton couples directly to an energetic line, one can always route its momentum such that the entire loop integral will depend only on $n_{i-}kn_{i+}^\mu/2$ of a single collinear direction, i , and no soft invariant can be formed to provide a scale to the loop diagram.

上述论证可以推广为如下全阶陈述: 与树 level 情况不同，在软定理的软圈修正中，辐射出的软引力子必须始终连接到纯软顶点，永远不直接连接到任何高能粒子线。原因是软共线相互作用中，在多极展开点 x_-^μ 的软场对 λ 展开是任意阶的。因此，如果辐射出的引力子直接耦合到一条高能线，我们总可以对它的动量作流道分配，使得整个圈积分仅依赖于单个共线方向的 $n_{i-}kn_{i+}^\mu/2$ ，即 i ，无法构造出软不变量为圈图提供标度。

Continuing with two soft loops, whenever the diagram contains a second purely soft vertex (as in the right-most diagram of Fig.9), there is another $\mathcal{O}\left(\lambda^2\right)$ suppression factor from this vertex, and the diagram can at best contribute to the sub-sub-leading soft factor, in agreement with the assertion. This conclusion can potentially be evaded by coupling the soft lines to the energetic lines with $\mathcal{O}\left(\lambda^0\right)$ or $\mathcal{O}\left(\lambda^1\right)$ vertices, which adds only linear eikonal-type propagators, possibly raised to higher powers due to the multipole expansion. The argument in the previous paragraph excludes coupling the emitted soft graviton to an energetic line (as in the left-most diagram of Fig. 9), but the middle two graphs in the figure are still an option. We next show that these graphs obtain another $\mathcal{O}\left(\lambda^2\right)$ suppression from the soft-collinear vertices on the energetic lines and they can only contribute to the sub-sub-leading term in the soft theorem consistent with the claim. [21]

接下来讨论双软圈: 只要图中包含第二个纯软顶点 (如图 9 最右侧的图所示)，这个顶点就会带来额外的 $\mathcal{O}\left(\lambda^2\right)$ 压低因子, 该图最多只能贡献次次领头软因子, 与我们的结论一致。如果软线通过 $\mathcal{O}\left(\lambda^0\right)$ 或 $\mathcal{O}\left(\lambda^1\right)$ 顶点耦合到高能线，本结论可能不成立——这种耦合仅添加线性程函型传播子，由于多极展开，传播子可能被升到更高次幂。上一段的论证排除了辐射出的软引力子耦合到高能线的情况 (对应图 9 最左侧的图)，但图中中间的两个图仍需要讨论。我们接下来说明，这两个图会从高能线上的软共线顶点得到额外的 $\mathcal{O}\left(\lambda^2\right)$ 压低，因此它们只能贡献软定理中的次次领头项，与我们的结论一致。[21]

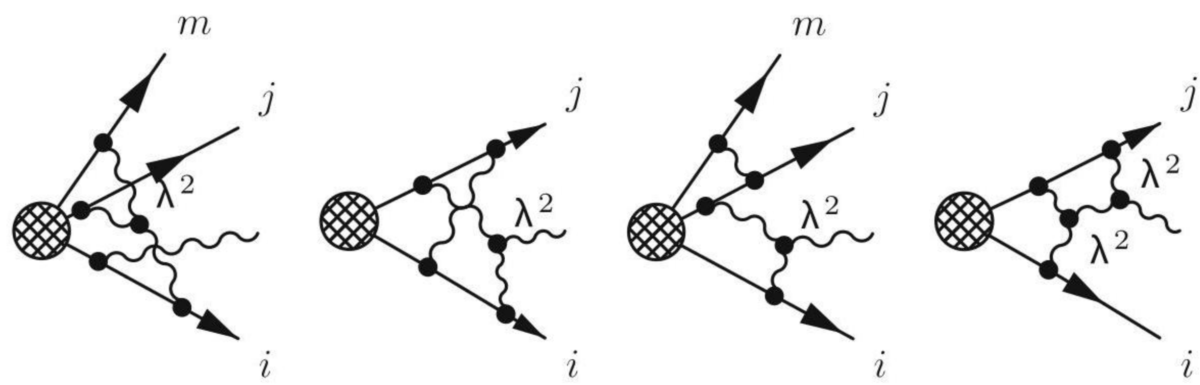

Fig. 9 Examples of soft two-loop diagrams. Left: Diagram with soft emission from a collinear line. Middle: Two diagrams with soft-collinear interactions and a purely soft vertex. Right: A nonvanishing soft two-

loop diagram. In order for the soft scale k to be present in both loops, the loops must be connected via purely soft interactions to the emitted graviton, which results in a contribution at sub-sub-leading order due to λ^4 suppression

图 9 双软圈图的示例。左: 软从共线线辐射出的图。中: 两个包含软共线相互作用和一个纯软顶点的图。右: 一个非零的双软圈图。要让两个圈中都存在软标度 k ，圈必须通过纯软相互作用连接到辐射出的引力子，由于 λ^4 压低，贡献出现在次次领头阶。

Consider first the case that all soft graviton couplings to the energetic particle lines are through the leading-power soft-collinear interaction $\frac{\kappa}{4} s_{--} \chi_c^\dagger \left(n_+ \partial^2 \chi_c\right)$ from (133). This gives rise to the eikonal interaction, which has the well-known property [42] that multiple emissions take the form of independent emissions after summing over all possible attachments to the energetic lines. For $i = 1, \ldots, N$ such lines with external momenta $\{p_i\}$, the emission of an arbitrary number of soft gravitons with momenta $\{q_k\}$ is given by the non-radiative amplitude multiplied by the eikonal factor:

首先考虑所有软引力子与高能粒子线的耦合都通过式 (133) 中的领头幂次软对撞相互作用 $\frac{\kappa}{4} s_{--} \chi_c^\dagger \left(n_+ \partial^2 \chi_c\right)$ 的情况。这会产生程函相互作用，它有一个众所周知的性质 [42]: 对高能线所有可能的连接求和后，多重辐射可表示为独立辐射的形式。对于外动量为 $\{p_i\}$ 的 $i = 1, \ldots, N$ 条此类粒子线，发射任意数量动量为 $\{q_k\}$ 的软引力子，振幅等于非辐射振幅乘以程函因子:

$$\mathcal{A}\left(\{p_i\}\right) \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{k=1}^{m} \left(\sum_{i=1}^{N} \frac{\kappa}{2} \frac{p_i^{\mu_k} p_i^{\nu_k} \varepsilon_{\mu_k \upsilon_k}(k)}{p_i \cdot q_k + i\varepsilon} \right). \tag{150}$$

To construct a loop diagram with a single emitted soft graviton, one must tie together the q_k among each other with a soft graviton propagator or with the purely soft vertex from which the external graviton is emitted. The key observation is that due to the factorized form of the denominator of the eikonal amplitude (150), the external momentum k does not enter the loop integral of those pairs of q_k tied together directly, which therefore can only depend on the energetic momenta p_i . Now, a dimensionally regulated soft loop integral must be proportional to $\left(S/\mu^2\right)^\varepsilon$, where S is a soft invariant, that is, a scalar product of momenta, which scales as λ^4 , and has mass dimension two. Since no such invariant can be built from the available momenta as $p_i \cdot p_j \sim \lambda^0$ and $p_i^2 = 0$, these integrals must be scaleless, and therefore, the entire graph vanishes. This applies to the two middle diagrams in Fig. 9 when all soft-collinear vertices are from $\mathcal{L}^{(0)}$, which would otherwise contribute at $\mathcal{O}\left(\lambda^2\right)$. Note that this vanishing also holds if some of the q_k are tied together through purely soft vertices, as long as the emitted graviton is not attached to them, since in this case k^μ never enters the corresponding subgraph. This results in the following all-order statement: If a loop integral involves only the leading-power eikonal interactions from $\mathcal{L}^{(0)}$, it vanishes unless all soft propagators are connected to the external graviton line through purely soft vertices. As a consequence, any non-vanishing soft n -loop graph has a suppression of at least $\mathcal{O}\left(\lambda^{2n}\right)$ from the combination of the required purely soft vertices.

要构造单软引力子发射的圈图，必须将 q_k 之间通过软引力子传播子，或是发射外引力子的纯软顶点连接起来。核心结论是: 由于程函振幅 (150) 的分母满足因子化形式，外动量 k 不会出现在直接相连的 q_k 对的圈积分中，因此这些积分仅依赖于能动量 p_i 。维度正规化的软圈积分必然正比于 $\left(S/\mu^2\right)^\varepsilon$ ，其中 S 是软不变量，即动量的标量积，标度为 λ^4 ，质量维度为 2。由于无法从现有动量 $p_i \cdot p_j \sim \lambda^0$ 和 $p_i^2 = 0$ 构造出这类不变量，这些积分必为无标度积分，因此整个图贡献为零。当所有软对撞顶点都来自 $\mathcal{L}^{(0)}$ 时，该结论适用于图 9 中的中间两个图，否则它们会在 $\mathcal{O}\left(\lambda^2\right)$ \$ 阶贡献。注意只要发射的引力子不连接在这些 q_k 上，即使部分 q_k 通过纯软顶点相连，贡献为零的结论依然成立，因为这种情况下 k^μ 绝不会进入对应的子图。由此可得全阶结论: 若圈积分仅包含来自 $\mathcal{L}^{(0)}$ 的领头幂次程函相互作用，则除非所有软传播子都通过纯软顶点连接到外引力子线，否则积分结果为零。因此，任何非零的软 n 圈图都至少会因所需纯软顶点的组合带来 $\mathcal{O}\left(\lambda^{2n}\right)$ 的压低。

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[21] This and the following arguments fill a loop hole in the discussion of soft loops in [15], where it is assumed that the dimensionful coupling κ that comes with a loop must be compensated by a soft momentum $k \sim \mathcal{O}(\lambda)$, so a k soft loop diagram must be $\mathcal{O}\left(\lambda^{2k}\right)$. However, in soft-collinear interactions, the factor of κ can be compensated by powers of the $\mathcal{O}(1)$ collinear momenta as there exists a $\mathcal{O}\left(\lambda^0\right)$ vertex. In other words, for soft-collinear interactions, the loop expansion in κ is not equivalent to a λ^2 -expansion.

[21] 本文及以下论证补上了文献 [15] 讨论软圈时的一个漏洞: 该文献假设圈图带来的量纲耦合 κ 必须由软动量 $k \sim \mathcal{O}(\lambda)$ 抵消，因此 k 阶软圈图必然是 $\mathcal{O}\left(\lambda^{2k}\right)$ 阶。但在软对撞相互作用中，κ 的因子可以由共线能动量的 $\mathcal{O}(1)$ 次幂抵消，因为存在 $\mathcal{O}\left(\lambda^0\right)$ 顶点。换言之，对软对撞相互作用而言，κ \$ 中的圈展开不等价于 λ^2 \$ 展开。

There is still the possibility that the emitted graviton couples to an energetic line through some of the sub-leading in λ soft-collinear interactions from $\mathcal{L}^{(1)}$ (110) and $\mathcal{L}^{(2)}$ (111) . The vertices from $\mathcal{L}^{(1)}$ contain a single transverse index. In the frame $p_{i\perp}^\mu = 0$, there is no transverse external vector available; hence, diagrams with only a single $\mathcal{L}^{(1)}$ insertion on an energetic line vanish. It already follows that one cannot evade the λ^2 suppression per loop by the use of sub-leading soft-collinear interactions and the two-loop diagrams are all at least of order $\mathcal{O}\left(\lambda^4\right)$. However, these diagrams actually vanish. At the two-loop order, the only nonvanishing topologies contain two purely soft vertices connected by a soft propagator as in the right-most diagram of Fig. 9 or a purely soft four-graviton vertex, which also scales at least as $\mathcal{O}\left(\lambda^4\right)$.

仍存在一种可能: 发射出的引力子通过 λ 软共线相互作用中的某些次领头阶耦合到高能线。来自 $\mathcal{L}^{(1)}$ (110) 和 $\mathcal{L}^{(2)}$ (111) 的顶点包含单个横向指标。在坐标系 $p_{i\perp}^\mu = 0$ 中，不存在可用的外部横向矢量；因此，高能线上仅插入单个 $\mathcal{L}^{(1)}$ 的图为零。由此已经可以得出: 无法利用次领头阶软共线相互作用来避免每个圈的 λ^2 压低，两圈图的阶数都至少为 $\mathcal{O}\left(\lambda^4\right)$ 。但实际上这些图都等于零。在两圈阶，唯一非零的拓扑包含两个由软传播子连接的纯软顶点 (如图 9 最右侧的图)，或是一个纯软四引力子顶点，其标度也至少为 $\mathcal{O}\left(\lambda^4\right)$ 。

To see this, assume that the single purely soft three-graviton vertex connects the emitted graviton to the energetic legs i and j . Since the sub-leading eikonal vertices that provide further $\mathcal{O}\left(\lambda^2\right)$ suppression reside on a single energetic leg, one can further assume without loss of generality that line i has only leading-power eikonal vertices attached to it and then route the external momentum k^μ through this energetic leg. Leg i is therefore an eikonal leg, that is, after summing over all permutations of attached momenta, the amplitude for this leg takes the form of independent emissions. All other external energetic lines also have this property, except for the one with sub-leading soft-collinear vertices, which may or may not be leg j . The situation and momentum assignments are illustrated in Fig. 10.

> 为此我们假设，单个纯软三引力子顶点将发射出的引力子连接到高能支 i 和 j 。由于提供额外 $\mathcal{O}(\lambda^2)$ 压低的次领头阶 eikonal 顶点位于单条高能支上，我们可以不失一般性地进一步假设，线 i 仅附着领头幂次 eikonal 顶点，再将外动量 k^μ 通过这条高能支传递。因此支 i 是 eikonal 支，即对所有附着动量排列求和后，该支的振幅可写为独立发射的形式。所有其他外部高能线也满足这一性质，唯一例外是带次领头阶软共线顶点的线，该线可以是 leg j ，也可以不是。这种情形与动量分配如图 10 所示。

At the two-loop order, there is only one pair of open lines to be tied up with a soft graviton propagator, but for later purposes, it is convenient to proceed with the general situation shown in the figure. Because leg i is an eikonal leg, the momenta k and q appear in the entire diagram only in denominators

> 两圈阶仅存在一对需要用软引力子传播子连接的开线，但为了后续讨论，按图中所示的一般情形处理更方便。由于 leg i 是 eikonal 支，动量 k 和 q 仅出现在全图的分母中

$$\frac{1}{(k+q)^2 q^2} \frac{1}{n_{i-} \cdot (k+q)} \times \prod_k \frac{1}{n_{j-} \cdot L_k\left(q, l_j\right)}, \tag{151}$$

where $L_k\left(q, l_j\right)$ is the linear sum of a subset (or all) of the momenta attached to leg j . Note that this expression is general, since it was not assumed that leg j is eikonal. Let l be one of the momenta l_j . The loop integral over l is then of the form

> 其中 $L_k\left(q, l_j\right)$ 是附着在支 j 上的部分 (或全部) 动量的线性和。注意该表达式是通用的，因为我们并未假设支 j 是 eikonal 支。设 l 是动量 l_j 之一，则对 l 的圈积分形式为

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{n_{a-} \cdot l} \prod_k \frac{1}{n_{j-} \cdot L_k\left(q, l_j\right)}, \tag{152}$$

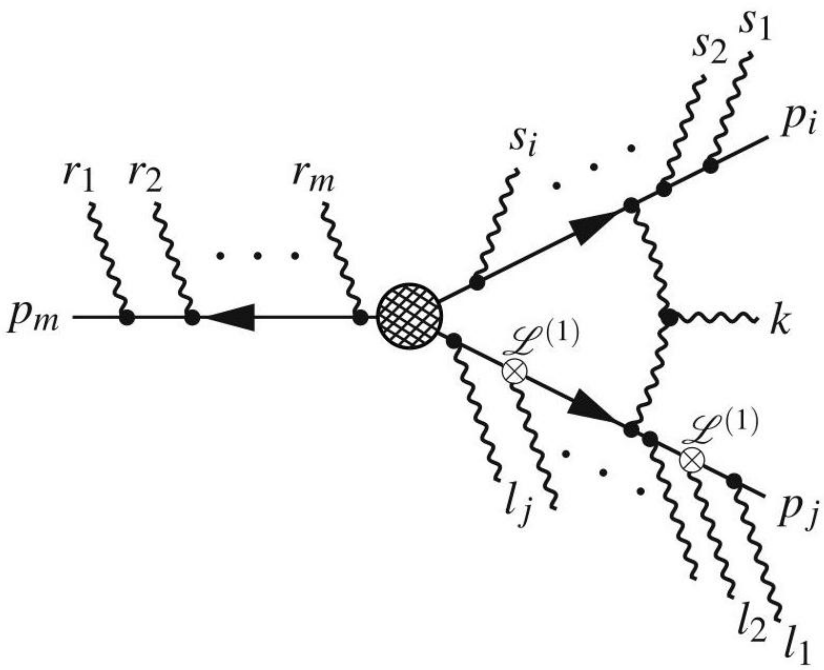

Fig. 10 $\mathcal{O}\left(\lambda^4\right)$ diagram with one purely soft vertex and sub-leading soft-collinear interactions on line j . Except for the emitted graviton with momentum k , all other wavy lines have to be tied together among each other

> 图 10 $\mathcal{O}\left(\lambda^4\right)$ 图，线上 j 存在一个纯软顶点和次领头阶软共线相互作用。除了动量为 k 的发射引力子外，所有其他波浪线都需要相互连接

where $a \neq j$ refers to the eikonal leg to which the graviton line l is attached and we used that the emissions are independent from this leg. After integrating over l , the result can only depend on $\left(n_{j-}q\right) n_{j+}^{\mu}/2$ and $\left(n_{j-}l_j\right) n_{j+}^{\mu}/2$, from which one cannot form a non-vanishing invariant, hence the integral is scaleless, and consequently the entire diagram is zero. If $a = j$, the expression (152) without the factor $1/\left(n_{a-} \cdot l\right)$ holds, and the same conclusion is reached.

> 其中 $a \neq j$ 指向引力子线 l 所附着的 eikonal 支，我们利用了发射过程相对于该支是独立的。对 l 积分后，结果仅能依赖于 $\left(n_{j-}q\right) n_{j+}^{\mu}/2$ 和 $\left(n_{j-}l_j\right) n_{j+}^{\mu}/2$ ，无法由此构造出非零不变量，因此该积分是无标度积分，整个图最终为零。若为 $a = j$ ，则不含因子 $1/\left(n_{a-} \cdot l\right)$ 的式 (152) 仍然成立，可得到相同结论。

To complete the proof of the main assertion for soft loop corrections, it remains to show that three- and higher-loop diagrams are inevitably more suppressed than $\mathcal{O}\left(\lambda^4\right)$, so that they cannot affect the soft theorem to the sub-sub-leading order. The preceding two all-order statements imply that a diagram that is not already excluded by them must have (1) exactly one purely soft vertex of $\mathcal{O}\left(\lambda^2\right)$ to which the external soft graviton is attached and (2) $\mathcal{O}\left(\lambda^2\right)$ suppression from sub-leading soft-collinear interactions on a single energetic leg, say j , plus (3) an arbitrary number of additional leading-power eikonal interactions. The property of independent soft emission therefore applies to all energetic lines except j . However, this case is covered by the arguments of the previous paragraph, which holds to any loop order.

为完成软圈修正主论断的证明，仍需证明三圈及更高圈图不可避免地比 $\mathcal{O}(\lambda^4)$ 压低更多，因此它们不会影响次次领头阶的软定理。前述两个全阶表述表明，未被二者排除的图必须满足:(1) 外部软引力子恰好耦合到 $\mathcal{O}(\lambda^2)$ 的一个纯软顶点，(2) 单个能动腿 (记为 j) 上的次领头软 collinear 相互作用给出 $\mathcal{O}(\lambda^2)$ 的压低，以及 (3) 任意数量额外的领头功率程光学相互作用。因此，除 j 外，独立软辐射的性质适用于所有能动线。不过前一段的论证已经涵盖了该情形，该论证对任意圈阶都成立。

Having established suppression of either only collinear or only soft loop corrections in accordance with the claim, we finally turn to arbitrary loop diagrams that can have both, which start from two loops. It is easy to see that two-loop diagrams must be at least $\mathcal{O}(\lambda^4)$, since the external soft graviton must still attach to a purely soft-collinear vertex by the same argument as before, while the collinear loop still costs a factor of λ^2 . The case of higher-loop diagrams is covered by the argument of the previous paragraph, which continues to hold when the leg with the sub-leading soft-collinear Lagrangian insertions is replaced by the leg to which a collinear loop is attached. Any mixed collinear-soft loop diagram with more than two collinear loops is already beyond the next-to-next-to-soft order and does not have to be considered. This concludes the characterization of the structure of loop corrections to the soft theorem, as formulated at the beginning of this subsection.

我们已经根据论断证明了仅共线或仅软圈修正的压低性质，最后来讨论从两圈开始、同时包含二者的任意圈图。不难看出，两圈图至少是 $\mathcal{O}(\lambda^4)$ 阶，因为沿用之前的论证，外部软引力子仍必须耦合到纯软共线顶点，而共线圈仍需要一个 λ^2 因子。更高圈图的情形已经被前一段的论证涵盖: 当次领头软共线拉氏量插入所在的腿被替换为耦合共线圈的腿时，该论证依然成立。任意包含超过两个共线圈的共线-软混合圈图都已经超出次次软阶，无需考虑。至此我们完成了本小节开头表述的软定理圈修正结构刻画。

The one- and two-loop corrections to the soft theorem are largely universal, since they arise almost exclusively from effective Lagrangian terms. The non-universal part can be parameterized in terms of the matching coefficients $\widetilde{C}^X(t_i)$ of SCET source operators with up to two additional collinear transverse graviton fields. A complete calculation of the loop-corrected soft factors has, however, not yet been performed. [22]

软定理的一圈和两圈修正具有很大的普遍性，因为它们几乎完全来源于有效拉氏量项。非普遍部分可以通过 SCET 源算符的匹配系数 $\widetilde{C}^X(t_i)$ 参数化，该算符最多包含两个额外的共线横引力子场。但目前尚未对圈修正后的软因子完成完整计算。[22]

Cross-References

交叉引用

Effective Field Theory and Applications

有效场论及其应用

- Quantum General Relativity and Effective Field Theory

  - 量子广义相对论与有效场论

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22 The leading logarithmically enhanced terms in the energy of the soft graviton for the emission from massive scalars have been computed in [37,38].

22 本文在文献 [37,38] 中计算了大质量标量粒子辐射出软引力子能量中领头对数增强项。

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